ON A GENERALIZATION OF LITTLEWOOD'S CONJECTURE

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Abstract. We present a class of lattices in $\mathbb{R}^d$ ($d \geq 2$) which we call *grid-Littlewood* lattices and conjecture that any lattice is such. This conjecture is referred to as GLC. Littlewood's conjecture amounts to saying that $\mathbb{Z}^2$ is grid-Littlewood. We then prove existence of grid-Littlewood lattices by first establishing a dimension bound for the set of possible exceptions. Existence of vectors (*grid-Littlewood* vectors) in $\mathbb{R}^d$ with special Diophantine properties is proved by similar methods. Applications to Diophantine approximations are given. For dimension $d \geq 3$ we give explicit constructions of grid-Littlewood lattices (and in fact lattices satisfying a much stronger property). We also show that GLC is implied by a conjecture of G. A. Margulis concerning bounded orbits of the diagonal group. The unifying theme of the methods is to exploit rigidity results in dynamics ([EKL],[B],[LW]), and derive results in Diophantine approximations or the geometry of numbers.

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1. Introduction

1.1. **Introductory discussion.** In this paper we wish to discuss a generalization of the following well known conjecture due to Littlewood (see [Ma1]):

Conjecture 1.1 (Littlewood)**.** *$\forall \alpha, \beta \in \mathbb{R}$, $\inf_{n\neq 0} |n| \langle n\alpha\rangle\langle n\beta\rangle = 0$, where for $\gamma \in \mathbb{R}$, we denote $\langle\gamma\rangle = \min_{n\in\mathbb{Z}} |\gamma - n|$.*

Our interpretation of this conjecture, which naturally leads to its generalization to be described bellow, is as follows: Let us denote by $N : \mathbb{R}^2 \to \mathbb{R}$ the function $(x, y)^t \mapsto x \cdot y$ (where t stands for transpose). Given a vector $v = (\alpha, \beta)^t \in \mathbb{R}^2$, and an integer n, the set of values N takes on the set $\mathbb{Z}^2 + nv \subset \mathbb{R}^2$, is $\{(n\alpha + k)(n\beta + \ell) : k, \ell \in \mathbb{Z}\}$. The distance of this set to zero is exactly $\langle n\alpha\rangle\langle n\beta\rangle$. Let us denote this distance by $N(\mathbb{Z}^2 + nv)$. It is clear that this quantity attains arbitrarily small values as n varies. Littlewood's conjecture asserts that the rate of decay is faster then n. To this end we see that Littlewood's conjecture could be restated as saying

$$\forall v \in \mathbb{R}^2, \inf_{n\neq 0} |n| N(\mathbb{Z}^2 + nv) = 0.$$

Adopting this view point we may ask if this should be a special property of the lattice $\mathbb{Z}^2$. We conjecture that in fact any lattice in the plane should satisfy a similar property (see conjecture 1.2 bellow). In this paper we shall use methods from homogeneous dynamics to establish the existence of lattices in the plane which satisfy the generalized Littlewood conjecture and connect this conjecture to the dynamics of the diagonal group on the space of lattices in the plane. The main tool is the deep measure classification theorem obtained by Einsiedler Katok and Lindenstrauss in [EKL]. Applications to Diophantine approximations are presented.

1.2. **Notation Results and conjectures.** We first fix our notation and define the basic objects to be discussed in this paper. Throughout this paper $d \geq 2$ is an integer. Let X_d denote the space of d-dimensional unimodular lattices in $\mathbb{R}^d$ (i.e. of covolume 1) and let Y_d denote the space of translates of such lattices. Points of Y_d will be referred to as *grids*, hence for $x \in X_d$ and $v \in \mathbb{R}^d$, $y = x + v \in Y_d$ is the grid obtained by translating the lattice x by the vector v. We denote by π the natural projection

$$Y_d \xrightarrow{\pi} X_d, \quad x + v \mapsto x. \tag{1.1}$$

In the next section we shall see that X_d, Y_d are homogeneous spaces of Lie groups and equip them with metrics. For the meantime, the reader should think of the points of X_d, Y_d simply as subsets of $\mathbb{R}^d$.
Let $N : \mathbb{R}^d \to \mathbb{R}$ denote the function $N(w) = \prod_1^d w_i$. For a grid $y \in Y_d$, we denote

$$N(y) = \inf\{|N(w)| : w \in y\}. \tag{1.2}$$

Caution: Our use of the symbol N is ambiguous from several respects. It denotes simultaneously functions defined on Y_d and $\mathbb{R}^d$ and moreover, the dimension, d, will vary from time to time in our discussion. We choose this notation to be consistent with notation in the literature.

For each $x \in X_d$, we identify the fiber $\pi^{-1}(x)$ in Y_d with the torus $\mathbb{R}^d/x$. This enables us to define an operation of multiplication by an integer n on Y_d, i.e. if $y = x + v \in Y_d$ then $ny = x + nv$. The main objective of this paper is to discuss the following generalization of Littlewood's conjecture, referred to in this paper as GLC:

Conjecture 1.2 (GLC). *For any $d \geq 2$ and $y \in Y_d$*

$$\inf_{n \neq 0} |nN(ny)| = 0. \tag{1.3}$$

Definition 1.3. *(1) A grid $y \in Y_d$ is Littlewood if* (1.3) *holds.*

(2) A lattice $x \in X_d$ is grid-Littlewood if any grid $y \in \pi^{-1}(x)$ is Littlewood.

(3) A vector $v \in \mathbb{R}^d$ is grid-Littlewood if for any $x \in X_d$, the grid $y = x + v$ is Littlewood.

Thus conjecture 1.2 could be rephrased as saying that any lattice (resp vector) is grid-Littlewood. Of particular interest are grid-Littlewood lattices and vectors. For example, as explained in the previous subsection, when $d = 2$, $x = \mathbb{Z}^2 \in X_2$ and $v = (\alpha, \beta)^t \in \mathbb{R}^2$, the grid $x + v$ satisfies (1.3), if and only if the numbers α, β satisfy Littlewood conjecture stated above (conjecture 1.1). Hence, in our terminology, Littlewood's conjecture states simply that *the lattice $\mathbb{Z}^2$ is grid-Littlewood*. In this paper we shall prove existence of both grid-Littlewood lattices and vectors in any dimension $d \geq 2$ and give applications to Diophantine approximations. As remarked above, the methods of proof are those of homogeneous dynamics. Let us now turn to describe the relevant group actions.

$SL_d(\mathbb{R})$ and its subgroups acts naturally, via the linear action on $\mathbb{R}^d$, on the spaces X_d, Y_d. The projection π from (1.1) commutes these actions. Of particular interest to us will be the action of the group A_d of $d \times d$ diagonal matrices with positive diagonal entries and determinant one. The action of A_d on $\mathbb{R}^d$ preserves $N : \mathbb{R}^d \to \mathbb{R}$ and in turn, $N : Y_d \to \mathbb{R}$ is A_d invariant too. As a consequence, the set of exceptions to GLC

$$\mathcal{E}_d = \{y \in Y_d : y \text{ is not Littlewood}\} \tag{1.4}$$

is A_d invariant. The main result in this paper is the following:

Theorem 1.4. *The set of exceptions to GLC, $\mathcal{E}_d$, is contained in a countable union of sets of upper box dimension $\leq \dim A = d - 1$.*

We remark that from the dimension point of view, this is the best possible result without actually proving GLC, because of the A_d invariance.

In the fundamental paper [EKL], Einsiedler Katok and Lindenstrauss proved that the set of exceptions to Littlewood's conjecture is a countable union of sets of upper box dimension zero. The main tool in their proof is a deep measure classification theorem. The proof of theorem 1.4 is based on the same ideas and techniques and further more goes along the lines of [EK]. The new ingredient in the proof is lemma 3.2. As corollaries of this we get:

Corollary 1.5. *The set of $x \in X_d$ (resp $v \in \mathbb{R}^d$) that are not grid-Littlewood, is contained in a countable union of sets of upper box dimension $\leq \dim A_d = d - 1$. In particular, almost any lattice (resp vector) is grid-Littlewood.*

Corollary 1.6 (cf [EKL] Theorem 1.5)**.** *For a fixed lattice $x \in X_d$, the set $\{y \in \pi^{-1}(x) : y$ is not Littlewood$\}$ is contained in a countable union of sets of upper box dimension zero.*

Corollary 1.7. *Any set in X_d which has positive dimension transverse to the A_d orbits, must contain a grid-Littlewood lattice. In particular if the dimension of the closure of an orbit $A_d x$ ($x \in X_d$) is bigger than $d-1$, then x is grid-Littlewood.*

Theorem 1.4 as well as its corollaries are proved in § 3. We remark here that the only proof we know for the existence of grid-Littlewood lattices in dimension 2 and for grid-Littlewood vectors of any dimension, goes through the proof of theorem 1.4. For lattices of dimension $d \geq 3$ the situation is different. As will be seen in § 5, for $d \geq 3$, one can exploit rigidity results on commuting hyperbolic toral automorphisms proved by Berend in [B], and give explicit constructions of grid-Littlewood lattice (and in fact of lattices which are *grid-Littlewood of finite type*, see definition 5.1).

1.3. **Applications to Diophantine approximations.** Recall that for $\eta \in \mathbb{R}$ we denote $\langle \eta \rangle = \min_{k \in \mathbb{Z}} |\eta + k|$. We denote by η^* the integer defined by the equation

$$\langle \eta \rangle = |\eta + \eta^*| .$$

We shall prove the following theorems in § 4

Theorem 1.8. *In any set $J \subset [0,1]$ of positive dimension, there is a number α with the following property: $\forall \beta, \gamma \in \mathbb{R}$, there exists a sequence $n_i \in \mathbb{Z}$ such that $|n_i| \to \infty$ and*

$$\lim \max \{\langle n_i\gamma \rangle ; \langle (n_i\gamma)^*\alpha + n_i\beta \rangle\} = 0, \quad \lim |n_i| \langle n_i\gamma \rangle \langle (n_i\gamma)^*\alpha + n_i\beta \rangle = 0. \tag{1.5}$$

In particular, when J is the set of badly approximable numbers in the unit interval and we choose $\beta = \alpha$ and let γ be arbitrary or choose $\gamma = \alpha$ and let β be arbitrary or even choose $\beta = \gamma = \alpha$ we get the following immediate corollary

Corollary 1.9.

1. *There are badly approximable numbers $\alpha \in [0,1]$ such that $\forall \gamma \in \mathbb{R}$ there exists a sequence $n_i \in \mathbb{Z}$ such that $|n_i| \to \infty$ and*

$$\lim \max \{\langle n_i\gamma \rangle ; \langle ((n_i\gamma)^* + n_i)\, \alpha \rangle\} = 0, \quad \lim |n_i| \langle n_i\gamma \rangle \langle ((n_i\gamma)^* + n_i)\, \alpha \rangle = 0. \tag{1.6}$$

2. *There are badly approximable numbers $\alpha \in [0,1]$ such that $\forall \beta \in \mathbb{R}$ there exists a sequence $n_i \in \mathbb{Z}$ such that $|n_i| \to \infty$ and*

$$\lim \max \{\langle n_i\alpha \rangle ; \langle (n_i\alpha)^*\alpha + n_i\beta \rangle\} = 0, \quad \lim |n_i| \langle n_i\alpha \rangle \langle (n_i\alpha)^*\alpha + n_i\beta \rangle = 0, \tag{1.7}$$

3. *There are badly approximable numbers $\alpha \in [0,1]$ such that there exists a sequence $n_i \in \mathbb{Z}$ such that $|n_i| \to \infty$ and*

$$\lim \max \{\langle n_i\alpha \rangle ; \langle ((n_i\alpha)^* + n_i)\, \alpha \rangle\} = 0, \quad \lim |n_i| \langle n_i\alpha \rangle \langle ((n_i\alpha)^* + n_i)\, \alpha \rangle = 0. \tag{1.8}$$

Theorem 1.10. *In any set of dimension more than 1 in the plane, there exists a vector $(\beta, \gamma)^t$ such that for any $\alpha \in \mathbb{R}$ there exists a sequence $n_i \in \mathbb{Z}$ with $|n_i| \to \infty$ such that (1.5) holds. In particular one could choose $\alpha = \beta$ or $\alpha = \gamma$.*

1.4. **Interaction with other conjectures.** We end this section by noting that GLC is implied by a conjecture of G.A.Margulis (see [Ma2]) which goes back to [CaSD].

Conjecture 1.11 (Margulis)**.** *Any bounded A_{d+1} orbit in X_{d+1} is compact.*

We prove at the end of §§2.2

Proposition 1.12. *Conjecture 1.11 implies GLC.*

Remark: As will be seen, given a lattice $x \in X_d$, the richer the dynamics of its orbit under A_d, the easier it will be to prove that it is grid-Littlewood. This suggests an explanation to the difficulty of proving that $\mathbb{Z}^2$ is grid-Littlewood (for it has a divergent orbit) as opposed to proving for example that a lattice with a compact or a dense orbit is grid-Littlewood (see Problem 5.13).

Acknowledgments: I would like express my deepest gratitude to my advisers, Hillel Furstenberg and Barak Weiss for their constant help and encouragement. Special thanks are due to Manfred Einsiedler for his significant contribution to the results appearing in this paper. I would also like to express my gratitude to the mathematics department at the Ohio State University and to Manfred Einsiedler, for their warm hospitality during a visit in which much of the research was conducted.

2. Preparations

2.1. X_d, Y_d **as homogeneous spaces.** Denote $G_d = SL_d(\mathbb{R}), \Gamma_d = SL_d(\mathbb{Z})$. We identify X_d with the homogeneous space G_d/Γ_d in the following manner: For $g \in G_d$, the coset $g\Gamma_d$ represents the lattice spanned by the columns of g. We denote this lattice by $\bar{g}$. Y_d is identified with $\left(G_d \ltimes \mathbb{R}^d\right) / \left(\Gamma_d \ltimes \mathbb{Z}^d\right)$ similarly, i.e. for $g \in G_d$ and $v \in \mathbb{R}^d$, the coset $(g, v)\Gamma_d \ltimes \mathbb{Z}^d$ is identified with the grid $\bar{g} + v$. We endow X_d, Y_d with the quotient topologies thus viewing them as homogeneous spaces. We define a natural embedding $\tau : Y_d \hookrightarrow X_{d+1}$ in the following manner: $\forall y = \bar{g} + v \in Y_d$,

$$\tau_y = \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \Gamma_{d+1} \in X_{d+1}. \tag{2.1}$$

Note that this embedding is proper. G_d and its subgroups act on X_d, Y_d by multiplication from the left. The reader should check that under our identifications, these actions of G_d agrees with the linear action on $\mathbb{R}^d$, when applied to points of X_d, Y_d, thought of as subsets of $\mathbb{R}^d$. We embed G_d in G_{d+1} (in the upper left corner) thus allowing G_d and its subgroups to act on X_{d+1} as well. Note that the action commutes with τ.

2.2. **Linking dynamics to GLC.** The following observation is useful in connection with GLC: $\forall y \in Y_d$

$$\inf_{n \neq 0} |nN(ny)| = \inf \left\{ |N(w)| : w \in \tau_y, w_{d+1} \neq 0 \right\}. \tag{2.2}$$

Note the multiple use of the symbol N in the above equation. On the left hand side N refers to a function on Y_d while on the right hand side, to a function on $\mathbb{R}^{d+1}$.

Proposition 2.1 (Inheritance)**.**

(1) If $y, y_0 \in Y_d$ are such that $\tau_{y_0} \in \overline{A_{d+1}\tau_y}$, then if y_0 is Littlewood then so is y.
(2) If $x, x_0 \in X_d$ are such that $x_0 \in \overline{A_d x}$, then if x_0 is grid-Littlewood then so is x.

Proof. The proof of (1) follows from (2.2). The proof (2) follows from (1) and the compactness of the fibers of π. □

As done in [EKL] we link GLC to the dynamics of the following cone in A_{d+1}:

$$A^+_{d+1} = \left\{ diag(e^{t_1} \dots e^{t_{d+1}}) \in A_{d+1} : t_i > 0, i = 1 \dots d \right\}. \tag{2.3}$$

Lemma 2.2. *For $y \in Y_d$, if $A^+_{d+1}\tau_y$ is unbounded (i.e has a noncompact closure), then y is Littlewood.*

Proof. Recall Mahler's compactness criterion that says that a set $C \subset X_{d+1}$ is bounded, if and only if there is a uniform positive lower bound on the lengths of non zero vectors belonging to points of C. Let us fix the supremum norm on $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$. Let $y \in Y_d$. Assume that in the orbit $A^+_{d+1}\tau_y$ there are lattices with arbitrarily short vectors. Given $0 < \epsilon < 1$, there exists $a \in A^+_{d+1}$ and $w \in \tau_y$ such that the vector aw is of length less than ϵ. In particular $N(aw) = N(w) < \epsilon$. We will be through by (2.2) once we justify that $w_{d+1} \neq 0$. Assume $w_{d+1} = 0$. It follows that the length of aw is greater than that of w, as A^+_{d+1} expands the first d coordinates. On the other hand, the vector $w' = (w_1 \dots w_d)^t \in \mathbb{R}^d$ (which has the same length as w) belongs to the lattice $\pi(y)$. Let ℓ denote the length of the shortest nonzero vector in $\pi(y)$. We obtain a contradiction once $\epsilon < \ell$. □

Proof of proposition 1.12. Given a grid $y \in Y_d$, if $A^+_{d+1}\tau_y$ is unbounded then by lemma 2.2 we know that y is Littlewood. Assume that $A^+_{d+1}\tau_y \subset K$ for some compact $K \subset X_{d+1}$. Choose any one parameter semigroup $\{a_t\}_{t\geq 0}$ in the cone A^+_{d+1} and let z be a limit point of the trajectory $\{a_t\tau_y : t \geq 0\}$ in K. We claim that z has a bounded A_{d+1} orbit. To see this note that for any $a \in A_{d+1}$ we have that for large enough t's, aa_t is in the cone A^+_{d+1}, thus $az \in K$. Assuming conjecture 1.11, we obtain that z has a compact A_{d+1} orbit. As τ_y contains vectors of the form $(*, \dots, *, 0)^t$ which can be made as short as we wish under the action of A_{d+1}, we see that the orbit $A_{d+1}\tau_y$ is unbounded in X_{d+1} by Mahler's compactness criterion. It follows that $z \notin A_{d+1}\tau_y$ and hence $z \in \overline{A_{d+1}\tau_y} \setminus A_{d+1}\tau_y$. Theorem 1.3 from [LW] states that any orbit closure of A_{d+1} in X_{d+1} which strictly contains a compact A_{d+1} orbit, is homogeneous. More precisely, there exists a closed group $H < G_{d+1}$, strictly containing A_{d+1} such that $\overline{A_{d+1}\tau_y} = Hz$. Such a group H must contain a group of the form $\{u_{ij}(t)\}_{t\in\mathbb{R}}$ for some $1 \leq i \neq j \leq d+1$, where $u_{ij}(t)$, is the unipotent matrix all of whose entries are zero but the diagonal entries which are equal to 1 and the ij'th entry that is equal to t. It is easy to see that for any $\epsilon > 0$ there exist some t such that $u_{ij}(t)z$ contains a vector v with $N(v) \in (\epsilon, 2\epsilon)$. Since $u_{ij}(t)z \in \overline{A_{d+1}\tau_y}$, we deduce that τ_y contains a vector w with $N(w) \in (\epsilon, 2\epsilon)$. We deduce that $w_{d+1} \neq 0$ and as ϵ was arbitrary, (2.2) implies that y is Littlewood as desired.

□

2.3. **Dimension and entropy.** Let us recall the notions of upper box dimension and topological entropy. Let (X,d) be a compact metric space. For any $\epsilon>0$ we denote by $\mathcal{S}_\epsilon=\mathcal{S}_\epsilon(X)$ the maximum cardinality of a set of points in X with the property that the distance between any pair of distinct points in it is greater or equal to ϵ (such a set is called $\epsilon-$ *separated*). We define the upper box dimension of X to be

$$\dim_{box} X=\limsup_{\epsilon\to 0}\frac{\log \mathcal{S}_\epsilon}{|\log\epsilon|}.$$

Since this is the only notion of dimension we will discuss, we shall denote it by $\dim X$. If we denote by $\mathcal{N}_\epsilon=\mathcal{N}_\epsilon(X)$ the minimum cardinality of a cover of X by sets of diameter less than ϵ, then we also have that $\dim X=\limsup_{\epsilon\to 0}\frac{\log\mathcal{N}_\epsilon}{|\log\epsilon|}$. Note that if $f:X\to Y$ is a bi-Lipschitz map, then $A\subset X$ is of zero dimension if and only if $f(A)\subset Y$ is.
If we have a continuous map $a:X\to X$, then for $\epsilon>0, n\in\mathbb{N}$ we denote by $\mathcal{S}_{n,\epsilon}=\mathcal{S}_{n,\epsilon}(X,a)$ the maximum cardinality of a set $S\subset X$ with the property that for any pair of distinct points $x,y\in S$ there exist some $0\le i\le n$ with $d(a^ix,a^iy)>\epsilon$ (such a set is called $(n,\epsilon)-$ *separated* for a). The topological entropy of a is defined to be

$$h_{top}(a)=\lim_{\epsilon\to 0}\limsup_{n}\frac{\log\mathcal{S}_{n,\epsilon}}{n}.$$

2.4. **Metric conventions and a technical lemma.** For a metric space (X,d) we denote by $B_\epsilon^X(p)$, the closed ball of radius ϵ around p. If X is a group $B_\epsilon^X=B_\epsilon^X(p)$ where p is the trivial element (zero or one according to the structure). Given Lie groups G,H... we denote their Lie algebras by the corresponding lower case Gothic letters $\mathfrak{g},\mathfrak{h}$... Let G be a Lie group. We choose a right invariant metric $d(\cdot,\cdot)$ on it, coming from a right invariant Riemannian metric. Let $\Gamma<G$ be a lattice in G. We denote the projection from G to the quotient $X=G/\Gamma$, by $g\mapsto\bar{g}$. We define the following metric on X (also denoted by $d(\cdot,\cdot)$)

$$d(\bar{g},\bar{h})=\inf_{\gamma_i\in\Gamma}d(g\gamma_1,h\gamma_2)=\inf_{\gamma\in\Gamma}d(g,h\gamma). \tag{2.4}$$

Under these metrics, for any compact set $K\subset X$ there exist an *isometry radius* $\epsilon(K)$, i.e. a positive number ϵ such that for any $x\in K$, the map $g\mapsto gx$ is an isometry between B_ϵ^G and $B_\epsilon^X(x)$. Given a decomposition of $\mathfrak{g}=\oplus_1^l V_i$, the map $v\mapsto\exp v_1\ldots\exp v_l$ (where $v=\sum v_i$ and $v_i\in V_i$) has the identity map as its derivative at zero. It follows that it is bi-Lipschitz on a ball of small enough radius around zero. We refer to such a map as a *decomposition chart* and to the corresponding radius as a *bi-Lipschitz radius*. When taking into account the above, we get that given a compact set $K\subset X$ and a decomposition $\mathfrak{g}=\oplus_1^l V_i$, we can speak of a *bi-Lipschitz radius* $\delta(K)$*, for* K *with respect to this decomposition chart*, i.e. we choose $\delta=\delta(K)$ to be small enough so that the image of $B_\delta^{\mathfrak{g}}$ under the decomposition chart will be contained in the ball of radius $\epsilon(K)$ around the identity element. Note that under these conventions a bi-Lipschitz radius for K with respect to a decomposition chart is always an isometry radius.

Let G be semisimple and $\mathbb{R}$-split (for our purpose it will be enough to consider $G=SL_d(\mathbb{R})$). Let $A<G$ be a maximal $\mathbb{R}$-split torus in G (for example the group of diagonal

matrices in $SL_d(\mathbb{R})$). We fix on $\mathfrak{g}$ a supremum norm with respect to a basis of $\mathfrak{g}$ whose elements belong to one dimensional common eigenspaces of the adjoint action of A. For an element $a \in A$ we denote by $U^{\pm}(a), \mathfrak{u}^{\pm}(a)$, the stable and unstable horospherical subgroups and Lie algebras associated with a. That is

$$U^{+}(a) = \{g \in G : a^{-n} g a^{n} \longrightarrow_{n\to\infty} e\}, U^{-}(a) = U^{+}(a^{-1}),$$
$$\mathfrak{u}^{\pm}(a) = \{X \in \mathfrak{g} : Ad_a^n(X) \longrightarrow_{n\to\mp\infty} 0\}.$$

We denote by $\mathfrak{u}^0(a)$ the Lie algebra of the centralizer of a, that is $\{X \in \mathfrak{g} : Ad_a(X) = X\}$. Note that from the semisimplicity of a, it follows that $\mathfrak{g} = \mathfrak{u}^+(a) \oplus \mathfrak{u}^0(a) \oplus \mathfrak{u}^-(a)$. When a fixed element $a \in A$ is given, we denote for $X \in \mathfrak{g}$, its components in $\mathfrak{u}^+, \mathfrak{u}^-, \mathfrak{u}^0$, by X^+, X^-, X^0, respectively.

We shall need the following lemma, the reader is advised to skip it for the time being and return to it after seeing it in use in the next section:

Lemma 2.3. *For a fixed element $e \neq a \in A$ there exist $\lambda > 1$ and $\delta, M, c > 0$ such that for any $X_i, Y_i \in B^{\mathfrak{g}}_{\delta}, i = 1, 2$ with $X_i \in \mathfrak{u}^+(a)$ and $||Y_1 - Y_2|| < \frac{||X_1 - X_2||}{M}$, if for an integer k, for any $0 \leq j \leq k$*

$$d(a^j \exp X_1 \exp Y_1, a^j \exp X_2 \exp Y_2) < \delta$$

then for any $0 \leq j \leq k$

$$d(a^j \exp X_1 \exp Y_1, a^j \exp X_2 \exp Y_2) \geq c\lambda^j ||X_1 - X_2||.$$

Proof. Let $\eta > 0$ be a bi-Lipschitz radius for the decomposition charts exp and φ, corresponding respectively to the trivial decomposition and the decomposition $\mathfrak{g} = \mathfrak{u}^+(a) \oplus \mathfrak{u}^0(a) \oplus \mathfrak{u}^-(a)$ i.e. $\varphi : B^{\mathfrak{g}}_{\eta} \to G$ is the map $\varphi(v) = \exp v^+ \exp v^0 \exp v^-$. Let $0 < \delta_1 < \eta$ satisfy

$$\forall X_i, Y_i \in B^{\mathfrak{g}}_{\delta_1}, i = 1, 2, \quad \exp X_1 \exp Y_1 \exp -Y_2 \exp -X_2 \in \varphi(B^{\mathfrak{g}}_{\eta}). \tag{2.5}$$

We can define $u : \left(B^{\mathfrak{g}}_{\delta_1}\right)^4 \to B^{\mathfrak{g}}_{\eta}$ by

$$\forall X_i, Y_i \in B^{\mathfrak{g}}_{\delta_1}, i = 1, 2 \quad u(X_i, Y_i) = \varphi^{-1}\left(\exp X_1 \exp Y_1 \exp -Y_2 \exp -X_2\right).$$

When $X_i, Y_i \in B^{\mathfrak{g}}_{\delta_1}, i = 1, 2$ are fixed, we simplify our notation and write instead of $u(X_i, Y_i), u^{\pm}(X_i, Y_i), u^0(X_i, Y_i)$, just $u, u^{\pm}, u^0$. Thus we have the identity: $\forall X_i, Y_i \in B^{\mathfrak{g}}_{\delta_1}, i = 1, 2$

$$\varphi(u) = \exp u^+ \exp u^0 \exp u^- = \exp X_1 \exp Y_1 \exp -Y_2 \exp -X_2. \tag{2.6}$$

Let us formulate two claims that we will use:

Claim 1: There exist $0 < \delta_2 < \delta_1$ and $0 < M, c_1$, such that

$$\forall X_i, Y_i \in B^{\mathfrak{g}}_{\delta_2}, i = 1, 2, \text{ with } X_i \in \mathfrak{u}^+, ||Y_1 - Y_2|| < \frac{||X_1 - X_2||}{M}$$
$$\text{we have } ||u^+|| > c_1 ||X_1 - X_2||. \tag{2.7}$$

Claim 2: There exist $0 < \delta_3 < \delta_2$, such that if $v \in B^{\mathfrak{g}}_{\eta}$ and $k \in \mathbb{N}$ are such that

$\forall 0\leq j\leq k, d\left(\varphi\left(Ad_a^j(v)\right),e\right)<\delta_3$, then $\forall 0\leq j\leq k, Ad_a^j(v)\in B_\eta^{\mathfrak{g}}$.

Let us describe how to conclude the lemma from these claims: Let λ be the minimum amongst the absolute values of the eigenvalues of Ad_a that are greater than 1. Choose $\delta=\delta_3$ as in *claim 2*, $M>0$ as in *claim 1* and $c=c_1\cdot c_2$, where c_1 is as in *claim 1* and c_2 satisfies $d(\varphi(v_1),\varphi(v_2))>c_2||v_1-v_2||$ for any $v_1,v_2\in B_\eta^{\mathfrak{g}}$. Note that because of the choice of the norm on $\mathfrak{g}$, for $v\in\mathfrak{g}$ one has for any integer j

$$||Ad_a^j(v)||=||Ad_a^j(v^+)+Ad_a^j(v^0)+Ad_a^j(v^-)||\geq||Ad_a^j(v^+)||\geq\lambda^j||v^+||. \tag{2.8}$$

Let X_i,Y_i and $k\in\mathbb{N}$ be as in the statement of the lemma. For any $0\leq j\leq k$ we have:

$$\begin{aligned}
\delta &> d(a^j\exp X_1\exp Y_1, a^j\exp X_2\exp Y_2)\\
&= d(a^j\exp X_1\exp Y_1\exp -Y_2\exp -X_2a^{-j},e)\\
&= d(a^j\varphi(u(X_i,Y_i))a^{-j},e)\\
&= d(\varphi(Ad_a^j(u)),e)\\
&> c_2||Ad_a^j(u)||\geq c_2\lambda^j||u^+||>c_1c_2\lambda^j||X_1-X_2||=c\lambda^j||X_1-X_2||.
\end{aligned}$$

We used the right invariance of the metric in the first equality, the fact that $\delta<\delta_1$ in the second and the relation $a\varphi(\cdot)a^{-1}=\varphi(Ad_a(\cdot))$ in the third. In the last row of inequalities we used *claim 2* and the choice of c_2 in the first inequality, (2.8) in the second and *claim 1* in the third. We now turn to the proofs of the above claims.

Notation 2.4. *If two positive numbers α,β, satisfy $r\alpha<\beta<\frac{1}{r}\alpha$, for some $r>0$, we denote it by $\alpha\sim_r\beta$.*

Proof of Claim 1. We use the notation of lemma 2.3. Let $0<\rho<\delta_1$ be such that the map $(v_1,v_2)\mapsto\exp v_1\exp -v_2$, takes $\left(B_\rho^{\mathfrak{g}}\right)^2$ into $\exp B_\eta^{\mathfrak{g}}$. Since η was chosen to be a bi-Lipschitz radius for the map exp, there is a smooth function $w:\left(B_\rho^{\mathfrak{g}}\right)^2\to B_\eta^{\mathfrak{g}}$ which satisfies the relation

$$\forall v_1,v_2\in B_\rho^{\mathfrak{g}},\quad \exp w(v_1,v_2)=\exp v_1\exp -v_2.$$

Note that if $v_1,v_2\in\mathfrak{u}^+$, then $w(v_1,v_2)\in B_\eta^{\mathfrak{u}^+}$. Let $X_i,Y_i\in B_\rho^{\mathfrak{g}}, i=1,2$. The expressions in (2.6) are equal to

$$\exp w(X_1,X_2)\exp Ad_{\exp X_2}w(Y_1,Y_2). \tag{2.9}$$

Let us sketch the line of proof we shall pursue: We show that $w(v_1,v_2)\sim_r||v_1-v_2||$, for some $r>0$. We choose the constants carefully in such a way that given X_i,Y_i as in the statement of the claim, then there exist a $v\in\mathfrak{g}$ of length less than half of $||X_1-X_2||$, with $\varphi(v)=\exp Ad_{\exp X_2}w(Y_1,Y_2)$. It then follows from (2.9) that

$$\exp u^+=\exp w(X_1,X_2)\exp v^+=\exp w\left(w(X_1,X_2),-v^+\right).$$

It then follows that (ignoring constants that will appear)

$$||u^+||=||w(X_1,X_2)+v^+||>||X_1-X_2||-||v^+||>\frac{||X_1-X_2||}{2}.$$

Let us turn now to the rigorous argument. The fact that η is a bi-Lipschitz radius for exp implies the existence of a constant $r>0$, such that $\forall v_1, v_2 \in B_\rho^{\mathfrak{g}}$

$$||v_1 - v_2|| \sim_r d(\exp v_1, \exp v_2) = d(\exp v_1 \exp -v_2, e) \sim_r ||w(v_1, v_2)||$$
$$\Rightarrow ||v_1 - v_2|| \sim_{r^2} ||w(v_1, v_2)||. \tag{2.10}$$

Let M_0 bound from above the operator norm of $Ad_{\exp v}$ as v ranges over $B_\rho^{\mathfrak{g}}$. In (2.9), we have

$$||Ad_{\exp X_2} w(Y_1, Y_2)|| \leq M_0 ||w(Y_1, Y_2)|| < \frac{M_0}{r^2}||Y_1 - Y_2||. \tag{2.11}$$

Let $0 < \delta_2 < \rho$ be such that $\frac{2\delta_2}{r^2} < \rho$. This implies by (2.10), that $\forall v_1, v_2 \in B_{\delta_2}^{\mathfrak{g}}$, $||w(v_1, v_2)|| < \rho$. There exist some $0 < \rho' < \eta$ such that $\exp(B_{\rho'}^{\mathfrak{g}}) \subset \varphi(B_\rho^{\mathfrak{g}})$. Note that from the fact that exp is bi-Lipschitz on $B_{\rho'}^{\mathfrak{g}}$ and φ^{-1} is bi-Lipschitz on $\exp\left(B_{\rho'}^{\mathfrak{g}}\right)$, it follows that there exist a constant $\tilde{r}$ such that

$$\forall w \in B_{\rho'}^{\mathfrak{g}}, ||w|| \sim_{\tilde{r}} ||\varphi^{-1}(\exp(w))||. \tag{2.12}$$

Let $M = \max\left\{\frac{2\delta_2 M_0}{r^2\rho'}, \frac{2M_0}{\tilde{r}r^4}\right\}$. It follows from (2.11), that if $X_i, Y_i \in B_{\delta_2}^{\mathfrak{g}}, i = 1, 2$ are such that $X_i \in \mathfrak{u}^+$ and $||Y_1 - Y_2|| < \frac{||X_1 - X_2||}{M}$, then

$$||Ad_{\exp X_2} w(Y_1, Y_2)|| < \frac{M_0||X_1 - X_2||}{r^2 M} \tag{2.13}$$

and by our choice of M

$$||Ad_{\exp X_2} w(Y_1, Y_2)|| < \rho'. \tag{2.14}$$

By the choice of ρ', there exist some $v \in B_\rho^{\mathfrak{g}}$ satisfying

$$\varphi(v) = \exp Ad_{\exp X_2} w(Y_1, Y_2). \tag{2.15}$$

Note that by (2.12) with $w = Ad_{\exp X_2} w(Y_1, Y_2)$, by (2.13), and by the choice of M

$$||v|| \sim_{\tilde{r}} ||Ad_{\exp X_2} w(Y_1, Y_2)|| \Rightarrow ||v|| < \frac{M_0||X_1 - X_2||}{\tilde{r}r^2 M} \leq \frac{r^2||X_1 - X_2||}{2}. \tag{2.16}$$

The expressions in (2.6) and in (2.9) are equal to

$$\exp u^+ \exp u^0 \exp u^- = \exp w(X_1, X_2) \exp v^+ \exp v^0 \exp v^-. \tag{2.17}$$

As remarked above, the fact that $X_i \in \mathfrak{u}^+$ implies that $w(X_1, X_2) \in \mathfrak{u}^+$. From our choice of ρ and the fact that $v^+, w(X_1, X_2) \in B_\rho^{\mathfrak{u}^+}$, it follows that $w\left(w(X_1, X_2), -v^+\right) \in \mathfrak{u}^+$ is defined and satisfies:

$$\exp w(X_1, X_2) \exp v^+ = \exp w\left(w(X_1, X_2), -v^+\right). \tag{2.18}$$

Because $w(\cdot, \cdot)$ takes values in $B_\eta^{\mathfrak{g}}$, and φ is injective on $B_\eta^{\mathfrak{g}}$, it follows from (2.17), (2.18), that

$$u^+ = w\left(w(X_1, X_2), -v^+\right).$$

Because of (2.10) and (2.16)

$$||u^+|| = ||w\left(w(X_1, X_2), -v^+\right)|| > r^2||w(X_1, X_2) + v^+||$$

$$\geq r^2 \left(||w(X_1, X_2)|| - ||v^+||\right) \geq r^4 ||X_1 - X_2|| - \frac{r^4}{2}||X_1 - X_2|| = \frac{r^4}{2}||X_1 - X_2||.$$

Thus *claim 1* follows with the above choices of δ_2 and M and with $c_1 = \frac{r^4}{2}$.
Proof of Claim 2. Let $M_1 = ||Ad_a||$. Let $0 < \delta_3 < \delta_2$ satisfy

$$B^G_{\delta_3} \subset \varphi\left(B^{\mathfrak{g}}_{\frac{\eta}{M_1}}\right). \tag{2.19}$$

Let $v \in B^{\mathfrak{g}}_\eta$ and $k \in \mathbb{N}$ satisfy the assumptions of *claim 2*. Assume by way of contradiction that there exist some $0 \leq j < k$ such that $Ad_a^j(v) \in B^{\mathfrak{g}}_\eta$ but $Ad_a^{j+1}(v) \notin B^{\mathfrak{g}}_\eta$. We conclude that

$$\eta < M_1 ||Ad_a^j(v)|| \Rightarrow Ad_a^j(v) \notin B^{\mathfrak{g}}_{\frac{\eta}{M_1}}.$$

This contradicts the assumption that $\varphi\left(Ad_a^j(v)\right) \in B^G_{\delta_3}$ and (2.19) because φ is injective on $B^{\mathfrak{g}}_\eta$. □

3. The set of exceptions to GLC

In this section we prove theorem 1.4 and its corollaries. We go along the lines of §4 in [EK] and the fundamental ideas appearing in [EKL]. The main hidden tool is the measure classification theorem in [EKL]. What prevents us from citing known results is the fact that in the embedding $\tau : Y_d \hookrightarrow X_{d+1}$ (2.1), the grids which are not Littlewood and thus have bounded A^+_{d+1} orbit, do not lie (locally) on single unstable leaves of elements in the cone, but lie on products of unstable leaves (see lemmas 3.1, 3.2).

Beginning of proof of theorem 1.4. Let M_i be an increasing sequence of compact sets in X_{d+1} such that $X_{d+1} = \cup_i M_i$. Denote $F_i = \left\{y \in Y_d : A^+_{d+1}\tau_y \subset M_i\right\}$. Lemma 2.2 implies that the set of exceptions to GLC satisfies

$$\mathcal{E}_d \subset \cup_i F_i. \tag{3.1}$$

F_i is a compact subset of Y_d. We shall prove that $\dim F_i \leq d-1$ and conclude the proof of the theorem. Denote $L_i = \tau(F_i)$. As τ is, locally, a bi-Lipschitz map, it will be enough to prove $\dim L_i \leq d-1$. As $L_i \subset \tau(Y_d)$, we wish to describe local neighborhoods in $\tau(Y_d)$ in a convenient way. To do this we take $\Omega \subset A_d$ to be a compact symmetric neighborhood of the identity and we denote

$$V_1 = \left\{ \begin{pmatrix} 0 & \dots & & & 0 \\ \star & \ddots & & & \vdots \\ \vdots & \ddots & & & \\ \star & \dots & \star & \ddots & \\ 0 & \dots & & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{d+1} \right\}, V_2 = \left\{ \begin{pmatrix} 0 & \star & & \dots & \star \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \star \\ 0 & \dots & & & 0 \end{pmatrix} \in \mathfrak{g}_{d+1} \right\}. \tag{3.2}$$

Given $\delta > 0$ and $x \in \tau(Y_d) \subset X_{d+1}$, the set $\Omega \exp B^{V_1}_\delta \exp B^{V_2}_\delta x$ is a compact neighborhood of x in $\tau(Y_d)$. As L_i is compact, we can cover it by finitely many sets of the form

$L_i \cap \Omega \exp B_\delta^{V_1} \exp B_\delta^{V_2} x$ (for a suitable choice of points $x \in L_i$). Now it is clear that for any $x \in L_i$, the intersection $L_i \cap \Omega \exp B_\delta^{V_1} \exp B_\delta^{V_2} x$, is contained in

$$\Omega \cdot \left(\exp B_\delta^{V_1} \exp B_\delta^{V_2} x \cap \left\{z \in X_{d+1} : A_{d+1}^+ z \subset \Omega M_i\right\}\right). \tag{3.3}$$

Thus, the theorem will follow once we prove that

$$\dim\left(\exp B_\delta^{V_1} \exp B_\delta^{V_2} x \cap \left\{z \in X_{d+1} : A_{d+1}^+ z \subset \Omega M_i\right\}\right) = 0, \tag{3.4}$$

as Ω is of dimension $d-1$. This is the content of corollary 3.3 bellow. In order to prove this corollary, we need to prove two lemmas. We will return and conclude the proof of theorem 1.4 after proving corollary 3.3.

The following two lemmas furnish the link between dimension and entropy in our discussion. Lemma 3.1 is essentially lemma 4.2 from [EK]. For the reader's convenience and the completeness of our presentation, we include the proof in the appendix. Lemma 3.2 is one of the new ingredients appearing in this paper. For convenience we shall use the following notation: Given a semigroup $C \subset A_{d+1}$ and a compact set $K \subset X_{d+1}$ we denote

$$K_C = \{x \in X_{d+1} : Cx \subset K\}. \tag{3.5}$$

Note that K_C is a compact (possibly empty) C-invariant set.

Lemma 3.1. *Let $C \subset A_{d+1}$ be a semigroup, $a \in C$ and $K \subset X_{d+1}$ a compact set. If for some $\delta > 0$ and $x \in K$*

$$\dim\left(\exp\left(B_\delta^{\mathfrak{u}^+(a)}\right) \cdot x \cap K_C\right) > 0$$

then a acts with positive topological entropy on K_C.

For the proof of theorem 1.4 we shall need the following generalization of lemma 3.1:

Lemma 3.2. *Let $C_2 \subset C_1 \subset A_{d+1}$ be semigroups, $a_i \in C_i, i = 1, 2,$ and $K \subset X_{d+1}$ a compact set. Assume that there exists subspaces V_i of $\mathfrak{u}^+(a_i)$ such that for any $b \in C_2$, $V_1 \subset \mathfrak{u}^-(b)$. Then, there exists $\delta > 0$, such that if for some $x \in K$*

$$\dim\left(\exp B_\delta^{V_1} \exp B_\delta^{V_2} \cdot x \cap K_{C_1}\right) > 0,$$

then either a_1 acts with positive topological entropy on K_{C_1}, or there exists a compact set $\tilde{K} \supset K$, such that a_2 acts with positive topological entropy on $\tilde{K}_{C_2}$.

The following corollary goes along the lines of Proposition 4.1 from [EK].

Corollary 3.3. *Let $C_2 \subset C_1 \subset A_{d+1}$ be open cones, $a_i \in C_i, i = 1, 2,$ and $K \subset X_{d+1}$ a compact set. Assume that there exists subspaces V_i of $\mathfrak{u}^+(a_i)$ such that for any $b \in C_2$, $V_1 \subset \mathfrak{u}^-(b)$. Then, there exists $\delta > 0$, such that for any $x \in K$*

$$\dim\left(\exp B_\delta^{V_1} \exp B_\delta^{V_2} \cdot x \cap K_{C_1}\right) = 0. \tag{3.6}$$

Proof. In the proof of Proposition 4.1 from [EK], it is shown that there cannot be an open cone $C \subset A_{d+1}$ that acts on a compact invariant subset of X_{d+1} such that some element in C acts with positive topological entropy. Taking δ to be as in lemma 3.2, we see that positivity of the dimension in (3.6) leads to a contradiction. $\square$

Remark: The highly non trivial part of the proof of theorem 1.4 is hidden in the proof of corollary 3.3. This is the use of the measure classification from [EKL].

Concluding the proof of theorem 1.4. In order to conclude the proof of the theorem we use corollary 3.3 with the following choices of K, C_i, V_i and a_i, $i = 1, 2$ to fit our needs in (3.4): We take V_1, V_2 to be as in (3.2) and

$$C_1 = A^+_{d+1}, \quad a_1 = diag\left(2, \ldots, d+1, 1/(d+1)!\right), \quad a_2 = diag\left(d+1, \ldots, 2, 1/(d+1)!\right).$$

Note that $V_2 = \mathfrak{u}(a_2)^+$. Moreover, $V_1 \subset \mathfrak{u}^-(a_2)$, thus we can choose C_2 to be an open cone containing a_2 and contained in C_1, such that for any $b \in C_2, V_1 \subset \mathfrak{u}^-(b)$. Finally we take $K = \Omega M_i$. □

Proof of lemma 3.2. Note that from the fact that $V_1 \subset \mathfrak{u}^-(a_2)$ it follows that the sum $V_1 + V_2$ is direct. Let V_3 be any subspace of $\mathfrak{g}_{d+1}$ such that $\mathfrak{g}_{d+1} = V_1 \oplus V_2 \oplus V_3$. Choose $\delta = \delta(K)$ to be a bi-Lipschitz radius for K with respect to the above decomposition (see §§ 2.4 for notation). Assume also that δ satisfies the conclusion of lemma 2.3 with $a = a_1$. For the sake of brevity we denote $B_i = B^{V_i}_\delta$. Assume $x \in K$ satisfies

$$\dim\left(\exp B_1 \exp B_2 x \cap K_{C_1}\right) > 0. \tag{3.7}$$

Since δ is a bi-Lipschitz radius for K (with respect to the decomposition $V_1 \oplus V_2 \oplus V_3$), (3.7) implies that the dimension of

$$F = F(\delta) = \{(X, Y) \in B_1 \times B_2 : \exp X \exp Y x \in K_{C_1}\}$$

is positive. From the choice of the norm on $\mathfrak{g}_{d+1}$ (see §§ 2.4) and from the assumption that for any $b \in C_2$, $V_1 \subset \mathfrak{u}^-(b)$, it follows that for any $X \in V_1$, $||Ad_b(X)|| \le ||X||$. Choose a compact set $\tilde{K} \supset \exp(B^{\mathfrak{g}_{d+1}}_\delta)K$. Denote by π_2 the projection from $B_1 \times B_2$ to B_2. There are two cases:

Case 1: Assume $\dim \pi_2(F) > 0$. We claim that $\exp\left(\pi_2(F)\right) x \subset \tilde{K}_{C_2}$. To see this, note that if $Y \in \pi_2(F)$ then there exists some $X \in B_1$ such that $\exp X \exp Y x \in K_{C_1}$, and so for any $b \in C_2$ we have

$$b \exp Y x = \exp Ad_b(-X) b \exp X \exp Y x \in \exp(B^{\mathfrak{g}_{d+1}}_\delta)K \subset \tilde{K}.$$

Now $\exp\left(\pi_2(F)\right) x \subset \exp B_2 \cdot x \cap \tilde{K}_{C_2}$ and therefore, positivity of the dimension of $\pi_2(F)$, implies the positivity of the dimension of $\exp B_2 \cdot x \cap \tilde{K}_{C_2}$. We apply lemma 3.1 and conclude that a_2 acts with positive topological entropy on $\tilde{K}_{C_2}$.

Case 2: Assume $\dim \pi_2(F) = 0$ and let us denote $\dim F = 3\rho$ with $\rho > 0$. We will show that a_1 acts with positive topological entropy on K_{C_1}. Recall the notation of lemma 2.3 (applied to a_1). We shall find for arbitrarily large integers n, finite sets $S_n \subset F$ with the following properties:

- For any pair of distinct points $(X_i, Y_i) \in S_n, i = 1, 2$
$$||X_1 - X_2|| > \lambda^{-n}, ||Y_1 - Y_2|| < \frac{\lambda^{-n}}{M}. \tag{3.8}$$
- $|S_n| > M^{-\rho}\lambda^{n\rho}$.

Given two distinct points in S_n, $(X_i, Y_i), i = 1, 2$, let us analyze the rate at which $\exp X_i \exp Y_i x$ drift apart from each other under the action of powers of a_1. For any $j \geq 0$ we have that $a_1^j \exp X_i \exp Y_i x \in K$ by the definition of F, and so, if the distance between these two points is less than δ (which is also an isometry radius for K), we have

$$\begin{aligned} &d(a_1^j \exp X_1 \exp Y_1 x, a_1^j \exp X_2 \exp Y_2 x) = \\ &d(a_1^j \exp X_1 \exp Y_1, a_1^j \exp X_2 \exp Y_2). \end{aligned} \tag{3.9}$$

By lemma 2.3, for any k such that for all $0 \leq j \leq k$, the expressions in (3.9) are smaller than δ, we have

$$d(a_1^j \exp X_1 \exp Y_1 x, a_1^j \exp X_2 \exp Y_2) \geq c\lambda^k ||X_1 - X_2|| > c\lambda^{k-n}.$$

In particular, if we set $\epsilon_0 = \min\{c, \delta\}$ then we must have some $0 \leq j \leq n$ for which

$$d(a_1^j \exp X_1 \exp Y_1 x, a_1^j \exp X_2 \exp Y_2 x) > \epsilon_0.$$

This means that $\{\exp X \exp Y x : (X, Y) \in S_n\}$ is an (n, ϵ_0)-separated set for (K_{C_1}, a_1). From here, it is easy to derive the positivity of the entropy by the bound we have on the size of S_n:

$$h_{top}(K_{C_1}, a_1) \geq \limsup \frac{1}{n} \log |S_n| \geq \lim_n \frac{1}{n} \log(M^{-\rho} \lambda^{n\rho}) = \rho \log \lambda > 0.$$

To build the sets S_n with the above properties, for arbitrarily large n's, we argue as follows: By definition of the dimension one can find a sequence $\epsilon_k \searrow 0$ such that

$$\mathcal{S}_{\epsilon_k}(F) > (1/\epsilon_k)^{2\rho}.$$

Choose $n_k \nearrow \infty$ such that $\lambda^{-n_k} \leq \epsilon_k < \lambda^{-n_k+1}$. It follows that

$$\mathcal{S}_{\lambda^{-n_k}}(F) \geq \mathcal{S}_{\epsilon_k}(F) > \lambda^{2n_k\rho - 2\rho}. \tag{3.10}$$

On the other hand, because we assume $\dim \pi_2(F) = 0$, for any large enough n

$$\frac{\log(\mathcal{N}_{\frac{\lambda^{-n}}{M}}(\pi_2(F)))}{\log(\lambda^n M)} < \rho.$$

Hence

$$\mathcal{N}_{\frac{\lambda^{-n}}{M}}(\pi_2(F)) < \lambda^{n\rho} M^{\rho}. \tag{3.11}$$

Denote $N_k = \mathcal{N}_{\frac{\lambda^{-n_k}}{M}}(\pi_2(F))$ and let $E_i^{(k)}, i = 1 \dots N_k$ be a covering of $\pi_2(F)$ by subsets of B_2 of diameter less than $\frac{\lambda^{-n_k}}{M}$. Since $N_k < \lambda^{n_k\rho} M^{\rho}$, by (3.10) and the pigeon hole principle, there must exist some $1 \leq i_k \leq N_k$ with

$$\mathcal{S}_{\lambda^{-n_k}}(\pi_2^{-1}(E_{i_k}^{(k)}) \cap F) > \lambda^{n_k\rho} M^{-\rho}. \tag{3.12}$$

Define S_{n_k} to be a maximal λ^{-n_k}-separated set in $\pi_2^{-1}(E_{i_k}^{(k)}) \cap F$. By construction, S_{n_k} has the desired properties. $\square$

Proof of corollary 1.5. The projection $\pi : Y_d \to X_d$ cannot increase dimension and therefor $\pi(\mathcal{E}_d)$, the set of lattices which are not grid-Littlewood is a countable union of sets of upper box dimension $\leq d-1$.
Denote by $p : X_d \times \mathbb{R}^d \to Y_d$ the map $p(x,v) = x+v$. It is bi-Lipschitz with a countable fiber and so if we denote by $p_2 : X_d \times \mathbb{R}^d \to \mathbb{R}^d$ the natural projection, then $p_2(p^{-1}(\mathcal{E}_d))$, the set of vectors which are not grid-Littlewood, is a countable union of sets of upper box dimension $\leq d-1$. □

Proof of corollary 1.6. Assume by way of contradiction that there exist $x \in X_d$ with

$$\dim\left(\pi^{-1}(x) \cap \mathcal{E}_d\right) > 0.$$

It follows that if $\Omega \subset A_d$ is a compact neighborhood of the identity, then

$$\Omega\left(\pi^{-1}(x) \cap \mathcal{E}_d\right),$$

has dimension greater than $\dim A_d = d-1$. A contradiction to theorem 1.4. □

Proof of corollary 1.7. Positivity of the dimension of a subset $L \subset X_d$, transverse to the A_d orbits, means that A_dL contains a compact set of dimension greater then $\dim A_d = d-1$. By theorem 1.4, such a set must contain a grid-Littlewood lattice. If the dimension of the A_d orbit closure of a lattice $x \in X_d$ is greater then $d-1$, then it contains a grid-Littlewood lattice. It now follows from proposition 2.1, that x is grid-Littlewood. □

Remark: The proof of theorem 1.4 gives a bit more than what is stated in its statement. It shows that the set

$$\left\{y \in Y_d : A_{d+1}^{+}\tau_y \text{ is bounded}\right\}$$

is a countable union of compact sets of upper box dimension $\leq d-1$. The corollaries of theorem 1.4 have corresponding versions as well.

4. Proof of theorems 1.8, 1.10

Proof of theorem 1.8. It follows from corollary 1.7 and the remark at the end of §3 that there exists $\alpha \in J$ such that the lattice x, spanned by the vectors $(1,0)^t, (\alpha,1)^t$ in the plane, is grid-Littlewood and moreover that for any grid $y \in \pi^{-1}(x)$ one has that $A_3^{+}\tau_y$ is unbounded in X_3. We now untie the definitions and translate this information to Diophantine information on α.
Given any vector $v = (\beta,\gamma)^t \in \mathbb{R}^2$, if we denote $y = x+v$, the fact that $A_3^{+}\tau_y$ is unbounded is equivalent (in light of Mahlerl's compactness criterion), to the existence of a sequence $a_i \in A_3^{+}$ going to infinity (i.e. leaving any compact subset) and a sequence of vectors $0 \neq w_i \in \tau_y$, such that $a_iw_i \to 0$ in $\mathbb{R}^3$. Denote

$$a_i = diag(e^{t_i}, e^{s_i}, e^{-(t_i+s_i)}); \quad w_i = (k_i + \ell_i\alpha + n_i\beta, \ell_i + n_i\gamma, n_i)^t \in \mathbb{R}^3,$$

where $k_i, \ell_i, n_i \in \mathbb{Z}$, not all equal to zero for each i. Then

$$\max\left\{e^{t_i}\left|k_i + \ell_i\alpha + n_i\beta\right|; e^{s_i}\left|\ell_i + n_i\gamma\right|; e^{-(t_i+s_i)}\left|n_i\right|\right\} \to 0. \tag{4.1}$$

As, $t_i, s_i \geq 0$, it follows from (4.1), first that $\ell_i = (n_i\gamma)^*$ and than that

$$\max\left\{\langle n_i\gamma\rangle; \langle (n_i\gamma)^*\alpha + n_i\beta\rangle\right\} \to 0. \tag{4.2}$$

Now, taking the product of the three quantities in (4.1) we get

$$|n_i|\,\langle n_i\gamma\rangle\langle (n_i\gamma)^*\alpha + n_i\beta\rangle \to 0. \tag{4.3}$$

We are left to justify why we can choose the n_i's so that $|n_i| \to \infty$. Note that if γ is rational and $1, \alpha, \beta$ are linearly dependent over $\mathbb{Q}$, then the statement of the theorem is clear. Assume that $|n_i|$ is bounded. We might as well assume its constant. Then ℓ_i is constant too and moreover $n_i\gamma = -\ell_i$, so γ is rational. It then follows from (4.2) that $1, \alpha, \beta$ are linearly dependent over $\mathbb{Q}$ which concludes the proof. □

Proof of theorem 1.10. From corollary 1.5 we get that any set in the plane which is of dimension > 1 must contain a grid-Littlewood vector $v = (\beta, \gamma)^t$. Moreover by the remark at the end of §3, we may assume that for any lattice $x \in X_2$, if we denote $y = x + v \in Y_2$, then $A_3^+\tau_y$ is unbounded in X_3. Given $\alpha \in \mathbb{R}$, we apply this to the lattice spanned by $(1,0)^t, (\alpha, 1)^t$ and continue as in the proof of theorem 1.8 above. □

5. Lattices that satisfy GLC

In this section we shall explicitly build grid-Littlewood lattices in $\mathbb{R}^d$ for $d \geq 3$. In fact these lattices will possess a much stronger property, namely:

Definition 5.1. *A grid $y \in Y_d$ is Littlewood of finite type if there exists a non zero integer n such that $N(ny) = 0$. A lattice $x \in X_d$ is grid-Littlewood of finite type if any $y \in \pi^{-1}(x)$ is Littlewood of finite type.*

Definition 5.2. *A grid $y \in Y_d$ is rational if y is an element of finite order in the group $\pi^{-1}(\pi(y)) = \mathbb{R}^d/\pi(y)$.*

The following list of observations is left to be verified by the reader.

Proposition 5.3.
1. *The set of Littlewood of finite type grids is A_d invariant.*
2. *If $y, y_0 \in Y_d$, $y_0 \in \overline{A_d y}$ and y_0 is Littlewood of finite type then y is Littlewood of finite type too.*
3. *If $x, x_0 \in X_d$, $x_0 \in \overline{A_d x}$ and x_0 is grid-Littlewood of finite type then x is grid-Littlewood of finite type too.*
4. *Any rational grid is Littlewood of finite type.*
5. *If $x_1 \in X_{d_1}$ is grid-Littlewood of finite type and $x_2 \in X_{d_2}$ is any lattice, then $x_1 \oplus x_2 \in X_{d_1+d_2}$ is grid-Littlewood of finite type.*
6. *If $x_1, x_2 \in X_d$ are such that x_1 is grid-Littlewood of finite type and there exist some $c > 0$ such that cx_1 is commensurable with x_2 then x_2 is grid-Littlewood of finite type.*
7. *The standard lattice $\mathbb{Z}^d$ is not grid-Littlewood of finite type. In fact, for any vector $v \in \mathbb{R}^d$, all of whose coordinates are irrationals we have that $\mathbb{Z}^d + v$ is not Littlewood of finite type.*

For $x \in X_d$, denote by $A_{d,x}$ its stabilizer in A_d. Note that $A_{d,x}$ acts on the torus $\pi^{-1}(x)$ as a group of automorphisms. From (2) and (4) of proposition 5.3, we deduce the following

Lemma 5.4. *If for any grid $y \in \pi^{-1}(x)$, $\overline{A_{d,x}y} \subset \pi^{-1}(x)$ contains an Littlewood of finite type grid then x is grid-Littlewood of finite type. In particular, if for any grid $y \in \pi^{-1}(x)$, $\overline{A_{d,x}y}$ contains a rational grid then x is grid-Littlewood of finite type.*

Recall that a group of automorphisms of a torus $\pi^{-1}(x)$ ($x \in X_d$) is called ID, if any infinite invariant set is dense. The following is a weak version of theorem 2.1 from [B]:

Theorem 5.5 (Theorem 2.1 [B]). *If the stabilizer $A_{d,x}$ of a lattice $x \in X_d$ under the action of A_d satisfies*

(1) There exist some $a \in A_{d,x}$ such that for any n the characteristic polynomial of a^n (which is necessarily over $\mathbb{Q}$) is irreducible.
(2) For each $1 \le i \le d$ there exist $a = diag(a_1 \dots a_d) \in A_{d,x}$ with $a_i \neq 1$.
(3) There exists $a_1, a_2 \in A_{d,x}$ which are multiplicatively independent (that is $a_1^k a_2^m = 1 \Rightarrow k = m = 0$).

then $A_{d,x}$ is ID.

We now turn to the construction of a family of grid-Littlewood of finite type lattices. Let K be a totally real number field of degree d over $\mathbb{Q}$. The ring of integers of K will be denoted by $\mathcal{O}_K$.

Definition 5.6. *(1) A lattice in K is the $\mathbb{Z}$-span of a basis of K over $\mathbb{Q}$.*
(2) If Λ is a lattice in K then its associated order is defined as $O_\Lambda = \{x \in K : x\Lambda \subset \Lambda\}$.

It can be easily verified that for any lattice Λ in K, O_Λ is a ring. Moreover, the units in this ring are exactly $O_\Lambda^* = \{\omega \in K : \omega\Lambda = \Lambda\}$. Dirichlet's unit theorem states the following

Theorem 5.7 (Dirichlet's unit theorem). *For any lattice Λ in K, the group of units O_Λ^* is isomorphic to $\{\pm 1\} \times \mathbb{Z}^{d-1}$.*

Let $\sigma_1 \dots \sigma_d$ be an ordering of the different embeddings of K into the reals. Define $\varphi : K \to \mathbb{R}^d$ to be the map whose i'th coordinate is σ_i. If we endow $\mathbb{R}^d$ with the structure of an algebra (multiplication defined coordinatewise), then φ becomes a homomorphism of $\mathbb{Q}$ algebras (here we think of the fields $\mathbb{Q}, \mathbb{R}$ as embedded diagonally in $\mathbb{R}^d$). It is well known that if Λ is a lattice in K, then $\varphi(\Lambda)$ is a lattice in $\mathbb{R}^d$. Let us denote by x_Λ the point in X_d obtained by normalizing the covolume of $\varphi(\Lambda)$ to be 1. We refer to such a lattice as a *lattice coming from a number field.* Because φ is a homomorphism

$$\varphi(O_\Lambda^*) \subset \left\{a \in \mathbb{R}^d : a x_\Lambda = x_\Lambda\right\}.$$

We can identify the linear map obtained by left multiplication by $a \in \mathbb{R}^d$ on $\mathbb{R}^d$ with the usual action of the diagonal matrix whose entries on the diagonal are the coordinates of a. We abuse notation and denote the corresponding matrix by the same symbol. After recalling that the product of all the different embeddings of a unit in an order equals ± 1 we get that in fact $\varphi(O_\Lambda^*)$ is a subgroup of the stabilizer of x_Λ in the group of diagonal matrices of determinant ± 1 (in fact there is equality here but we will not use it). To get back into SL_d we replace O_Λ^* by the subgroup $O_{\Lambda,+}^*$ of totally positive units (that is units, all of whose embeddings are positive). It is a subgroup of finite index in O^*_Λ. We conclude that φ will map $O^*_{\Lambda,+}$ into A_{d,x_Λ} (using our identification of vectors and diagonal matrices).

Lemma 5.8. *If $x_\Lambda \in X_d$ is a lattice coming from a totally real number field K of degree $d \geq 3$, then A_{d,x_Λ} is an ID group of automorphisms of $\pi^{-1}(x_\Lambda)$.*

Proof. It is enough to check that conditions (1),(2),(3) from theorem 5.5 are satisfied. Condition (2) is trivial. Condition (3) is a consequence of Dirichlet's units theorem and the assumption $d \geq 3$. To verify condition (1) we argue as follows: We will show that there exist $\alpha \in O^*_K$ such that for any n, α^n generates K (this is enough because $O^*_{\Lambda,+}$ is of finite index in O^*_K). Let $F_1 \dots F_k$ be a list of all the subfields of K. If we denote for a subset $B \subset K$

$$\sqrt{B} = \{x \in K : \exists n \text{ such that } x^n \in B\}$$

then we need to show that

$$O^*_K \setminus \cup_1^k \sqrt{O^*_{F_i}} \neq \emptyset. \tag{5.1}$$

Fix a proper subfield F of K. Note that the following is an inclusion of groups $O^*_F \subset \sqrt{O^*_F} \subset O^*_K$. Thus, Dirichlet's units theorem will imply (5.1) once we prove that O^*_F is of finite index in $\sqrt{O^*_F}$. We shall give a bound on the order of elements in the quotient $\sqrt{O^*_F}/O^*_F$ thus showing that the groups are of the same rank. It is enough to show that there exist some integer n_0 such that if $x \in K$ satisfies $x^n \in F$ for some n then $x^{n_0} \in F$. Let $x \in K$ be such an element. Denote by $\sigma_1 \dots \sigma_r$ the different embeddings of F into the reals and for any $1 \leq i \leq r$, denote by $\sigma_{ij}, j = 1 \dots s$ the different extensions of σ_i to an embedding of K into the reals. Thus $d = rs$ and σ_{ij} are all the different embeddings of K into the reals. Note that $x^n \in F$ if and only if for any $1 \leq i \leq r$ $\sigma_{i1}(x^n) = \dots = \sigma_{is}(x^n)$ i.e. if and only if $(\frac{\sigma_{ij}(x)}{\sigma_{ik}(x)})^n = 1$ for all i, j, k. But since there is a bound on the order of roots of unity in K we are done. ☐

We are now in position to prove

Theorem 5.9. *Any lattice coming from a totally real number field of degree $d \geq 3$ is grid-Littlewood of finite type.*

Proof. Let $x_\Lambda \in X_d$ be a lattice coming from a totally real number field of degree $d \geq 3$. Using lemma 5.8 and lemma 5.4 we see that the theorem will follow if we will show that any finite A_{d,x_Λ} invariant set in $\pi^{-1}(x_\Lambda)$ contain only rational grids. Assume that $y \in \pi^{-1}(x_\Lambda)$ lies in a finite invariant set. It follows that there exist $e \neq a \in \varphi(O^*_{\Lambda,+})$ with

$$ay = y. \tag{5.2}$$

Write $x_\Lambda = c\varphi(\Lambda)$, $y = x_\Lambda + v$ and $a = \varphi(\omega)$. Then from (5.2) it follows that there exist $\theta \in \Lambda$ such that in the algebra $\mathbb{R}^d$

$$v(\varphi(\omega) - 1) = c\varphi(\theta) \Rightarrow v = c\varphi(\theta(\omega - 1)^{-1}).$$

Since K is spanned over $\mathbb{Q}$ by Λ we see that v is in the $\mathbb{Q}$ span of $c\varphi(\Lambda) = x_\Lambda$ and hence y is a rational grid as desired. ☐

As a corollary of the ergodicity of the A_d action on X_d and proposition 5.3 (3), we get the following (we refer the reader to [Sh] for a stronger result).

Corollary 5.10. *Almost any $x \in X_d$ is grid-Littlewood of finite type for $d \geq 3$.*

The following result appears for example in [LW]:

Theorem 5.11. *The compact orbits for A_d in X_d are exactly the orbits of lattices coming from totally real number fields of degree d.*

This gives us the following corollary, which can be derived from proposition 5.3 (3), combined with theorem 5.9. We state it separately because of its interesting resemblance to theorem 1.3 from [LW].

Corollary 5.12. *For $x \in X_d$ ($d \geq 3$), if $\overline{A_d x}$ contains a compact A_d orbit, then x is grid-Littlewood of finite type.*

The reader is referred to the paper [Sh] where a significant strengthening of this corollary is established. Let us end this paper with two open problems which emerge from our discussion.

Problem 5.13. *Give an explicit example of a Littlewood lattice in dimension 2. In particular, prove that any lattice with a compact A_2 orbit, is Littlewood.*

Problem 5.14. *Does there exists two dimensional grid-Littlewood of finite type lattices?*

6. Appendix

Proof of lemma 3.1. Let the notation be as in lemma 3.1. The statement of lemma 2.3 simplifies when one chooses the Y_i's to be zero in the original statement:
Lemma 2.3, simplified version: *For a fixed element $e \neq a \in A_{d+1}$ there exist $\lambda > 1$ and $\eta, c > 0$ such that for any $X_i \in B_\eta^{\mathfrak{u}^+(a)}, i = 1, 2,$ if for an integer k, for any $0 \leq j \leq k$*

$$d(a^j \exp X_1, a^j \exp X_2) < \eta$$

then for any $0 \leq j \leq k$

$$d(a^j \exp X_1, a^j \exp X_2) \geq c\lambda^j ||X_1 - X_2||.$$

We apply this lemma for the element $a \in A_{d+1}$ appearing in the statement of lemma 3.1. Let $0 < \delta' < \max\{\eta, \delta\}$ be a bi-Lipschitz radius for K, with respect to the chart exp (see §§ 2.4 for notation). Cover the compact set $\exp B_\delta^{\mathfrak{u}^+(a)} x \cap K_C$ by finitely many sets of the form $\exp B_{\delta'}^{\mathfrak{u}^+(a)} y_i \cap K_C$, for a suitable choice of points $y_i \in K_C$. By assumption there exists an i such that $\dim \left(\exp B_{\delta'}^{\mathfrak{u}^+(a)} y_i \cap K_C\right) > 0$. Because δ' is a bi-Lipschitz radius, we have that the dimension of

$$F = F(\delta') = \left\{X \in B_{\delta'}^{\mathfrak{u}^+(a)} : \exp X y_i \in K_C\right\} \tag{6.1}$$

is positive. Denote it by 2ρ. By definition of dimension this means that there exists a sequence $\epsilon_k \searrow 0$, and $\epsilon_k - separated$ sets $S_k \subset F$, such that $|S_k| > \epsilon_k^{-\rho}$. Let $n_k \nearrow \infty$ be a sequence such that $\lambda^{-n_k} \leq \epsilon_k < \lambda^{-n_k+1}$. Let $X_1, X_2 \in S_k$ be two distinct points. Because δ' is also an isometry radius for K, if ℓ is an integer such that $\forall 0 \leq j \leq \ell$, $d(a^j \exp X_1 y_i, a^j \exp X_2 y_i) < \delta'$, then the simplified version of lemma 2.3, stated above implies that $\forall 0 \leq j \leq \ell$

$$\begin{aligned} \delta' &> d\left(a^j \exp X_1 y_i, a^j \exp X_2 y_i\right) \\ &= d\left(a^j \exp X_1, a^j \exp X_2\right) \\ &\geq c\lambda^j ||X_1 - X_2|| \geq c\lambda^j \epsilon_k > c\lambda^{j-n_k}. \end{aligned} \tag{6.2}$$

This means that if $\epsilon_0 = \min\{\delta', c\}$, then $\{\exp X y_i : X \in S_k\}$, is an $(\epsilon_0, n_k) - separating$ set for the action of a on K_C. We conclude that

$$h_{top}(K_C, a) \geq \lim_k \frac{1}{n_k} \log |S_k| \geq \lim_k \frac{-\rho \log \epsilon_k}{n_k} \geq \rho \log \lambda > 0.$$

Thus we achieve the desired conclusion. □

References

[B] D. Berend, *Multi-invariant sets on tori.* Trans. A.M.S . 280 (1983), 509-532.

[CaSD] J. W. S. Cassels and H. P. F. Swinnerton-Dyer, *On the product of three homogeneous linear forms and the indefinite ternary quadratic forms*, Philos. Trans. Roy. Soc. London. Ser. A. 248, (1955). 7396.

[EK] M. Einsiedler, D. Kleinbock, *Measure rigidity and p-adic Littlewood-type problems.* Compos. Math. 143 (2007), no. 3, 689–702.

[EKL] M. Einsiedler, A. Katok, and E. Lindenstrauss. *Invariant measures and the set of exceptions to Littlewoods conjecture.* Ann. of Math. (2), 164(2):513560, 2006.

[LW] E. Lindenstrauss, B. Weiss, *On sets invariant under the action of the diagonal group.* Ergodic Theory Dynam. Systems 21 (2001), no. 5, 1481–1500.

[Ma1] G. A. Margulis, *Oppenheim conjecture*, Fields Medallists lectures, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publishing, River Edge, NJ, 1997, pp. 272 327.

[Ma2] G. A. Margulis. *Problems and conjectures in rigidity theory*, Mathematics: frontiers and perspectives, pages 161-174. Amer. Math. Soc., Providence, RI, 2000.

[Sh] U. Shapira. *A solution to a problem of Cassels and Diophantine properties of cubic numbers.* Preprint.