CLOSED ORBITS FOR THE DIAGONAL GROUP AND WELL-ROUNDED LATTICES

MICHAEL LEVIN, URI SHAPIRA, AND BARAK WEISS

Abstract. Curt McMullen showed that every compact orbit for the action of the diagonal group on the space of lattices contains a well-rounded lattice. We extend this to all closed orbits.

1. Introduction

Let $n \geq 2$ be an integer, let $G \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{R})$, let $A \subset G$ be the subgroup of diagonal matrices with positive entries and let $\mathcal{L}_n \stackrel{\text{def}}{=} G/\mathrm{SL}_n(\mathbb{Z})$ be the space of unimodular lattices in $\mathbb{R}^n$. The dynamics of the A-action on $\mathcal{L}_n$ is a well-studied topic in view of applications to number theory. For instance, McMullen [McM05] studied this action in connection with his fundamental work on Minkowski's conjecture. A lattice $x \in \mathcal{L}_n$ is called *well-rounded* if the nonzero vectors of shortest length in x span $\mathbb{R}^n$, and McMullen proved that any compact A-orbit contains a well-rounded lattice. We show:

Theorem 1.1. *Suppose $x \in \mathcal{L}_n$ and Ax is closed. Then Ax contains a well-rounded lattice.*

The proof of Theorem 1.1 closely follows McMullen's strategy; namely McMullen deduced theorem 1.1 from a covering result regarding covers of the torus $\mathbb{T}^n$, while we deduce it from a different covering result.

McMullen also showed that any *bounded* A-orbit contains a well-rounded lattice *in its closure.* It is natural to inquire whether this result could be strengthened, by removing either of the italicized phrases. As we show in Proposition 2.2, this strengthening holds for almost every A-orbit, but whether or not it holds for *every* A-orbit is an open question. Our result could be seen as a partial step in this direction. Our proofs rely on results of Tomanov and the authors [TW03, SWb] classifying closed orbits for the A-action, as well as a covering result which is of independent interest. For another perspective on this and related questions, see [PS].

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2. Preliminaries

2.1. **The behavior of almost every lattice.** Let $\mathcal{WR} \subset \mathcal{L}_n$ denote the set of well-rounded lattices. Suppose $x \in \mathcal{WR}$ and $v_1, \ldots, v_n$ are linearly independent shortest vectors of x. We will say that x is a *generic well-rounded lattice* if for any $v \in x \smallsetminus \{0, \pm v_1, \ldots, \pm v_n\}$, $\|v\| > \|v_i\|$. For instance the lattice $\mathbb{Z}^n$ is generic well-rounded. We have:

Proposition 2.1. If $x \in \mathcal{L}_n$ is a generic well-rounded lattice, then there is an open $\mathcal{U} \subset \mathcal{L}_n$ such that $\mathcal{WR} \cap \mathcal{U}$ is a submanifold of $\mathcal{L}_n$ of codimension $n-1$.

Proof. The fact that x is generic implies that there is a neighborhood $\mathcal{V}$ of the identity in G, such that for $g \in \mathcal{V}$, any shortest vector of gx is gv_i for some i. Making $\mathcal{V}$ smaller if necessary we obtain that the multiplication map $g \mapsto gx$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{U} = \mathcal{V}x$, and we obtain that $\mathcal{WR} \cap \mathcal{U} = \{gx : \|gv_1\| = \cdots = \|gv_n\|\}$. Since $\{g \in G : \|gv_1\| = \cdots = \|gv_n\|\}$ is a subvariety of G cut out by $n-1$ independent equations, $\mathcal{WR} \cap \mathcal{U}$ is a submanifold of codimension $n-1$. □

Proposition 2.2. For almost every $x \in \mathcal{L}_n$, the orbit Ax contains a well-rounded lattice.

Proof. Let x_0 be a generic well-rounded lattice, with shortest vectors $v_1, \ldots, v_n$, and let $\mathcal{U}$ be the neighborhood of x_0 as in Proposition 2.1. Suppose in addition that derivatives of the maps $a \mapsto \|av_i\|$, $i = 1, \ldots, n-1$ are linearly independent (seen as linear functionals on the Lie algebra $\mathfrak{a}$). A simple computation shows that this condition is satisfied for the lattice $x_0 = \mathbb{Z}^n$. This condition implies that the orbit Ax_0 and the manifold $\mathcal{WR} \cap \mathcal{U}$ intersect transversally at x_0. In particular there is a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ such that if $x \in \mathcal{U}_0$ then there is $a \in A$ such that $ax \in \mathcal{WR}$. Thus any A-orbit entering $\mathcal{U}_0$ contains a well-rounded lattice. By Moore's ergodicity theorem (see e.g. [Zim84]) the A-action on $\mathcal{L}_n$ is ergodic, and hence almost every A-orbit enters $\mathcal{U}_0$. □

2.2. **The structure of closed orbits.** The following statement was proved by the authors in [SWb, Cor. 5.8], using earlier results of [TW03].

Proposition 2.3. Suppose Ax is closed. Then there is a decomposition $A = T_1 \times T_2$ and a direct sum decomposition $\mathbb{R}^n = \bigoplus_1^d V_i$ such that the following hold:

- Each V_i is spanned by some of the standard basis vectors.
- T_1 is the group of linear transformations which act on each V_i by a homothety, preserving Lebesgue measure on $\mathbb{R}^n$. In particular $\dim T_1 = d - 1$.
- T_2 is the group of diagonal matrices whose restriction to each V_i has determinant 1.
- $T_2 x$ is compact and $T_1 x$ is divergent; i.e. $Ax \cong T_1 \times T_2/(T_2)_x$, where $(T_2)_x \stackrel{\text{def}}{=} \{a \in T_2 : ax = x\}$ is cocompact in T_2.
- Setting $\Lambda_i \stackrel{\text{def}}{=} V_i \cap x$, each Λ_i is a lattice in V_i, so that $\bigoplus \Lambda_i$ is of finite index in x.

2.3. **Some preparations.** Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the standard basis of $\mathbb{R}^n$. For $1 \le d \le n$, let

$$\mathbf{I}_d^n \stackrel{\text{def}}{=} \{1 \le i_1 < \cdots < i_d \le n\}$$

denote the collection of multi-indices of length d and for $J = (i_1, \ldots, i_d) \in \mathbf{I}_d^n$ let $\mathbf{e}_J \stackrel{\text{def}}{=} \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_d}$ be the corresponding vector in the d-th exterior power of $\mathbb{R}^n$. We equip $\bigwedge^d \mathbb{R}^n$ with the inner product with respect to which $\{\mathbf{e}_J\}$ is an orthonormal basis, and denote by $\mathcal{E}_{d,n}$ the quotient of $\bigwedge^d \mathbb{R}^n$ by the equivalence relation $w \sim -w$. Note that the product of an element of $\mathcal{E}_{d,n}$ with a positive scalar is well-defined. We will (somewhat imprecisely) refer to elements of $\mathcal{E}_{d,n}$ as vectors. Given a subspace $L \subset \mathbb{R}^n$ with $\dim L = d$, we denote by $w_L \in \mathcal{E}_{d,n}$ the image of a vector of norm one in $\bigwedge^d L$. If $\Lambda \subset \mathbb{R}^n$ is a discrete subgroup of rank d, we denote by $w_\Lambda \in \mathcal{E}_{d,n}$ the image of the vector $v_1 \wedge \cdots \wedge v_d$, where $\{v_i\}_1^d$ forms a basis for Λ. The reader may verify that these vectors are well-defined (i.e. independent of the choice of the v_j) and satisfy $w_\Lambda = |\Lambda| w_L$ where $L = \operatorname{span} \Lambda$ and $|\Lambda|$ is the covolume of Λ in L, with respect to the volume form induced by the standard inner product on $\mathbb{R}^n$. We denote the natural action of G on $\mathcal{E}_{d,n}$ arising from the d-th exterior power of the linear action on $\mathbb{R}^n$, by $(g, w) \mapsto gw$. Given a subspace $L \subset \mathbb{R}^n$ and a discrete subgroup Λ we set

$$A_L \stackrel{\text{def}}{=} \{a \in A : aw_L = w_L\} \text{ and } A_\Lambda \stackrel{\text{def}}{=} \{a \in A : aw_\Lambda = w_\Lambda\}.$$

Note that $A_L = A_\Lambda$ when Λ spans L. Note also that the requirement $aw_L = w_L$ is equivalent to saying that $aL = L$ and $\det(a|_L) = 1$. Given a flag

$$\mathscr{F} = \{0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_k \subsetneq \mathbb{R}^n\} \tag{2.1}$$

(not necessarily full), let $A_{\mathscr{F}} \stackrel{\mathrm{def}}{=} \bigcap_i A_{L_i}$. The *support* of an element $w \in \mathcal{E}_{d,n}$ is the subset of $\mathbf{I}_d^n$ for which the corresponding coefficients of an element of $\bigwedge^d \mathbb{R}^n$ representing w are nonzero, and we write $\mathrm{supp}(L)$ or $\mathrm{supp}(\Lambda)$ for the supports of w_L and w_Λ. For $J = (i_1 < \cdots < i_d) \in \mathbf{I}_d^n$, set $\mathbb{R}^J \stackrel{\mathrm{def}}{=} \mathrm{span}\,(\mathbf{e}_{i_j})$ and define the multiplicative characters

$$\chi_J : A \to \mathbb{R}^*, \ \ \chi_J(a) \stackrel{\mathrm{def}}{=} \det(a|_{\mathbb{R}^J}).$$

Then for any subspace $L \subset \mathbb{R}^n$,

$$A_L = \bigcap_{J \in \mathrm{supp}(L)} \ker \chi_J \tag{2.2}$$

(and similarly for discrete subgroups Λ).

We fix an invariant metric $\mathbf{d}$ on A. We will need the following lemma (cf. [McM05, Theorem 6.1]):

Lemma 2.4. *Let $T \subset A$ be a closed subgroup and let $x \in \mathcal{L}_n$ be a lattice with a compact T-orbit. Then for any $C > 0$ there exists $R > 0$ such that for any collection $\{\Lambda_i\}$ of subgroups of x, there exists $b \in A$ such that*

$$\{a \in T : \forall i, \ \|a w_{\Lambda_i}\| \le C\} \subset \{a \in A : \mathbf{d}\,(a,\, b\,(\cap_i A_{\Lambda_i})) \le R\}\,. \tag{2.3}$$

Proof. In the argument below we will sometimes identify A with its Lie algebra $\mathfrak{a}$ via the exponential map, and think of the subgroups A_Λ as subspaces. By (2.2) only finitely many subspaces arise as A_Λ. In particular, given a collection of discrete subgroups $\{\Lambda_i\}$, the angles between the spaces they span (if nonzero) are bounded below, by a bound which is independent of the $\{\Lambda_i\}$. Therefore there exists a function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(R) \to_{R\to\infty} \infty$, such that

$$\begin{aligned} &\left\{a \in A : \forall\ J \in \cup_i \mathrm{supp}(w_{\Lambda_i}),\ \psi(R)^{-1} \le \chi_J(a) \le \psi(R)\right\} \subset \\ &\{a \in A : \mathbf{d}(a, \cap_i A_{\Lambda_i}) \le R\}\,. \end{aligned} \tag{2.4}$$

Since Tx is compact, there exists a compact subset $\Omega \subset T$ such that for any $a \in T$ there exists $b = b(a) \in T$ satisfying $bx = x$ and $b^{-1}a \in \Omega$. It follows that there exists $M \ge 1$ such that:

(I) for any subspace L, $||bw_L|| \le M||aw_L||$.
(II) for any multi-index J, $\chi_J(ba^{-1}) \le M$.

Given $C > 0$, let $C' \stackrel{\mathrm{def}}{=} MC$ and consider the finite set

$$\mathscr{S} \stackrel{\mathrm{def}}{=} \{\Lambda \subset x : ||w_\Lambda|| \le C'\}\,.$$

For any $\Lambda \in \mathscr{S}$ write $w_\Lambda = \sum_{J \in \mathrm{supp}(w_\Lambda)} \alpha_J(\Lambda)\mathbf{e}_J$. Let

$$0 < \varepsilon < \min\left\{|\alpha_J(\Lambda)| : \Lambda \in \mathscr{S}, J \in \mathrm{supp}(w_\Lambda)\right\},$$

and choose R large enough so that $\psi(R) > C'/\varepsilon$. We claim that for any $\{\Lambda_i\} \subset \mathscr{S}$,

$$\{a \in T : \forall i \ \|aw_{\Lambda_i}\| \le C\} \subset \{a \in T : \mathbf{d}(a, \cap_i A_{\Lambda_i}) \le R\}. \tag{2.5}$$

To prove this claim, suppose a is an element on the left hand side of (2.5). By (2.4) it is enough to show that for any $J \in \cup_i \operatorname{supp}(\Lambda_i)$ we have $\psi(R)^{-1} \le \chi_J(a) \le \psi(R)$. Since the coefficient of $\mathbf{e}_J$ in the expansion of aw_{Λ_i} is $\chi_J(a)\alpha_J(\Lambda_i)$ and since $||aw_{\Lambda_i}|| \le C$, we have

$$\chi_J(a) \le \frac{C}{|\alpha_J(\Lambda_i)|} \le \frac{C}{\varepsilon} \le \psi(R).$$

On the other hand, letting $b = b(a)$ we have $b\Lambda_i \in \mathscr{S}$ from (I), and

$$\begin{aligned} \varepsilon \le |\alpha_J(b\Lambda_i)| = \chi_J(b)|\alpha_J(\Lambda_i)| \implies \chi_J(b^{-1}) \le C/\varepsilon \\ \overset{\text{(II)}}{\implies} \chi_J(a^{-1}) = \chi_J(a^{-1}b)\chi_J(b^{-1}) \le C'/\varepsilon \le \psi(R), \end{aligned}$$

which completes the proof of (2.5).

Let $\{\Lambda_i\}$ be any collection of subgroups of x and assume that the set on the left hand side of (2.3) is non-empty. That is, there exists $a_0 \in T$ such that for all i, $||a_0 w_{\Lambda_i}|| \le C$. Let $b = b(a_0) \in T$, and set $\Lambda_i' \overset{\text{def}}{=} b\Lambda_i$. It follows that $\{\Lambda_i'\} \subset \mathscr{S}$ and so

$$\begin{aligned} \{a \in T : \forall i \, \|aw_{\Lambda_i}\| \le C\} &= b\left\{a \in T : \forall i \left\|aw_{\Lambda_i'}\right\| \le C\right\} \\ &\overset{(2.5)}{\subset} b\left\{a \in T : \mathbf{d}(a, \cap_i A_{\Lambda_i'}) \le R\right\} \\ &= \{a \in T : \mathbf{d}(a, b\,(\cap_i A_{\Lambda_i})) \le R\}, \end{aligned}$$

where in the last equality we used the fact that $A_{\Lambda_i'} = A_{\Lambda_i}$ because A is commutative. □

Lemma 2.5. *Let $\mathscr{F}$ be a flag of length k as in (2.1) and let $A_{\mathscr{F}}$ be its stabilizer. Then $A_{\mathscr{F}}$ is of co-dimension $\ge k$ in A.*

Proof. Given a nested sequence of multi-indices $J_1 \subsetneq \cdots \subsetneq J_k$ it is clear that the subgroup

$$\bigcap_{i=1}^{k} \ker \chi_{J_i}$$

is of co-dimension k in A. In light of (2.2), it suffices to prove the following claim:

Let $\mathscr{F}$ be a flag as in (2.1) with $d_i \overset{\text{def}}{=} \dim L_i$. Then there is a nested sequence of multi-indices $J_i \in \boldsymbol{I}^n_{d_i}$ such that $J_i \in \operatorname{supp}(L_i)$.

In proving the claim we will assume with no loss of generality that the flag is complete. Let $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$ such that $L_i =$

span $\{v_j\}_{j=1}^i$ for $i = 1, \ldots, n-1$. Let S be the $n \times n$ matrix whose columns are $v_1, \ldots, v_n$. Given a multi-index J of length $|J|$, we denote by S_J the square matrix of dimension $|J|$ obtained from S by deleting the last $n - |J|$ columns and the rows corresponding to the indices not in J. Note that with this notation, each w_{L_d} is the image in $\mathcal{E}_{d,n}$ of a vector proportional to

$$v_1 \wedge \cdots \wedge v_d = \sum_{J \in \mathbf{I}_d^n} (\det S_J)\mathbf{e}_J. \tag{2.6}$$

In particular, $J \in \operatorname{supp}(L_d)$ if and only if $\det S_J \neq 0$.

Proceeding inductively in reverse, we construct the nested sequence J_d by induction on $d = n, \ldots, 1$. Let $J_n = \{1, \ldots, n\}$ so that $S = S_{J_n}$. Suppose we are given multi-indices $J_n \supset \cdots \supset J_{d+1}$ such that $J_i \in \operatorname{supp}(w_{L_i})$ for $i = n, \ldots, d+1$. We want to define now a multi index $J_d \in \operatorname{supp}(w_{L_d})$ which is contained in J_{d+1}. By (2.6), $\det S_{J_{d+1}} \neq 0$. When computing $\det S_{J_{d+1}}$ by expanding the last column we express $\det S_{j_{d+1}}$ as a linear combination of $\{\det S_J : J \subset J_{d+1}, |J| = d\}$. We conclude that there must exist at least one multi-index $J_d \subset J_{d+1}$ for which $\det S_{J_d} \neq 0$. In turn, by (2.6) this means that $J_d \in \operatorname{supp}(w_{L_d})$. This finishes the proof of the claim. $\square$

3. Reduction to a topological statement

We will require the following topological result which generalizes Theorem 5.1 of [McM05]. Let s, t be non-negative integers, and let Δ denote the s-dimensional simplex, which we think of concretely as $\operatorname{conv}(\mathbf{e}_1, \ldots, \mathbf{e}_{s+1})$, where the $\mathbf{e}_j$ are the standard basis vectors in $\mathbb{R}^{s+1}$. We will discuss covers of $M \stackrel{\text{def}}{=} \Delta \times \mathbb{R}^t$, and give conditions guaranteeing that such a cover must cover a point at least $s + t + 1$ times. For $j = 1, \ldots, s+1$ let F_j be the face of Δ opposite to $\mathbf{e}_j$, that is $F_j = \operatorname{conv}(\mathbf{e}_i : i \neq j)$. Also let $M_j \stackrel{\text{def}}{=} F_j \times \mathbb{R}^t$ be the corresponding subset of M. Given $R > 0$ and a positive integer k, we say that a subset $U \subset \mathbb{R}^t$ is (R, k)-*almost affine* if it is contained in an R-neighborhood of a k-dimensional affine subspace of $\mathbb{R}^t$.

Theorem 3.1. *Suppose that $\mathcal{U}$ is a cover of M by open sets satisfying the following conditions:*

(i) For any connected component U of any element of $\mathcal{U}$ there exists j such that $U \cap M_j = \varnothing$.

(ii) There is R so that for any connected component U of the intersection of $k \leq s + t$ distinct elements of $\mathcal{U}$, the projection of U to $\mathbb{R}^t$ is $(R, s + t - k)$-almost affine.

Then there is a point of M which is covered at least $s+t+1$ times.

The case $s=0$ is McMullen's result, and the case $t=0$ is known as the Knaster-Kuratowski-Mazurkiewicz theorem (see e.g. [Kar]). Note that hypothesis (ii) is trivially satisfied when $k \le s$, since any subset of $\mathbb{R}^t$ is $(1,t)$-almost affine. We will prove Theorem 3.1 in §4. In this section we use it to prove Theorem 1.1.

Given a lattice $x \in \mathcal{L}_n$ let $\alpha(x)$ denote the length of a shortest nonzero vector in x. Given $\delta > 0$ let

$$\begin{aligned} \mathrm{Min}_\delta(x) &\stackrel{\mathrm{def}}{=} \{v \in x \smallsetminus \{0\} : \|v\| < (1+\delta)\alpha(x)\} \\ \mathbf{V}_\delta(x) &\stackrel{\mathrm{def}}{=} \operatorname{span} \mathrm{Min}_\delta(x) \\ \dim_\delta(x) &\stackrel{\mathrm{def}}{=} \dim \mathbf{V}_\delta(x). \end{aligned}$$

Finally, for $\varepsilon > 0$, let $\mathcal{U}^{(\varepsilon)} = \left\{U_j^{(\varepsilon)}\right\}_{j=1}^n$ be the collection of open subsets of A defined by

$$U_j = U_j^{(\varepsilon)} \stackrel{\mathrm{def}}{=} \{a \in A : \text{for all } \delta \text{ in a neighborhood of } j\varepsilon,\ \dim_\delta(ax) = j\}. \tag{3.1}$$

Note that these sets depend on $x \in \mathcal{L}_n$ but in our application x will be considered fixed and so we suppress this dependence from our notation. By [McM05, Thm. 7.2], $\mathcal{U}^{(\varepsilon)}$ is an open cover of A.

Lemma 3.2. *For any n there is a compact $K \subset \mathcal{L}_n$ such that if $x \in \mathcal{L}_n$ and $a \in U_n^{(\varepsilon)}$ for $\varepsilon < 1$, then $ax \in K$.*

Proof. If $a \in U_n^{(\varepsilon)}$ and $\varepsilon < 1$ then ax has n linearly independent vectors of length at most $(n+1)\alpha(ax)$. Since ax is unimodular, there is a constant C (depending only on n) such that $\alpha(x) \ge C$. The set $K \stackrel{\mathrm{def}}{=} \{x \in \mathcal{L}_n : \alpha(x) \ge C\}$ is compact by Mahler's compactness criterion, and fulfills the requirements. □

Proof of Theorem 1.1. It suffices to show that for each $\varepsilon > 0$, $U_n^{(\varepsilon)} \neq \varnothing$. Indeed, if this is the case, then letting ε_j be a sequence of positive numbers such that $\varepsilon_j \to 0$, for each j, we let $a_j \in U_n^{(\varepsilon_j)}$. By Lemma 3.2, the lattices $a_j x$ belong to a fixed compact set K, so there is subsequence converging to some $x_0 \in K$; we continue to denote the subsequence by $(a_j x)$. For each j there are linearly independent $v_1^{(j)}, \dots, v_n^{(j)} \in x$ such that $\left\|a v_i^{(j)}\right\| \le (1+\varepsilon_j)\alpha(a_j x)$. The angle between each $a_j v_i^{(j)}$ and the space spanned by the other $a_j v_\ell^{(j)}, \ell \neq i$ is bounded from below independently of j. Passing to a subsequence we can assume that each $a_j v_i^{(j)}$ converges to a nonzero vector $v_i \in x_0$. Since α is a continuous function, the v_i all have length equal to $\alpha(x_0)$, and by the lower bound on the angles between them, they are linearly independent; that is, x_0 is well-rounded.

In order to prove that $U_n^{(\varepsilon)}$ is non-empty, we will apply Theorem 3.1. The first step is to find a decomposition $A \simeq \mathbb{R}^{n-1} = \mathbb{R}^s \times \mathbb{R}^t$ and a simplex $\Delta \subset \mathbb{R}^s$, so that the restriction of the cover to $\Delta \times \mathbb{R}^t$ satisfies the two hypotheses of Theorem 3.1.

Let $A = T_1 \times T_2$ and $\mathbb{R}^n = \bigoplus_1^d V_i$ be the decompositions as in Proposition 2.3, and let $s \stackrel{\text{def}}{=} \dim T_1 = d-1$. For $a \in T_1$ we denote by $\chi_i(a)$ the number satisfying $av = e^{\chi_i(a)}v$ for all $v \in V_i$. Thus each χ_i is a homomorphism from T_1 to the additive group of real numbers. The mapping $a \mapsto \bigoplus_i \chi_i(a)\mathrm{Id}_{V_i}$, where Id_{V_i} is the identity map on V_i, is nothing but the logarithmic map of T_1 and it endows T_1 with the structure of a vector space. In particular we can discuss the convex hull of subsets of T_1. For each ρ we let

$$\Delta_\rho \stackrel{\text{def}}{=} \{a \in T_1 : \max_i \chi_i(a) \le \rho\}.$$

Then $\Delta_\rho = \mathrm{conv}(b_1, \ldots, b_d)$ where b_i is the diagonal matrix acting on each V_j, $j \neq i$ by multiplication by e^ρ, and contracting V_i by the appropriate constant ensuring that $\det b_i = 1$.

Let $P_i : \mathbb{R}^n \to V_i$ be the natural projection associated with the decomposition $\mathbb{R}^n = \bigoplus V_i$. Since $\bigoplus \Lambda_j$ is of finite index in x, each $P_i(x)$ contains Λ_i as a subgroup of finite index and hence is discrete in V_i. Moreover, the orbit $T_2 x$ is compact, so for each $a \in T_2$ there is a' belonging to a bounded subset of T_2 such that $ax = a'x$. This implies that there is $\eta > 0$ such that for any i and any $a \in T_2$, if $v \in ax$ and $P_i(v) \neq 0$ then $\|P_i(v)\| \geq \eta$. Let $C > 0$ be large enough so that $\alpha(x') \le C$ for any $x' \in \mathcal{L}_n$. Let ρ be large enough so that

$$e^\rho \eta > 2C. \tag{3.2}$$

We restrict the covers $\mathcal{U}^{(\varepsilon)}$ (where $\varepsilon \in (0, 1/n)$) to $\Delta_\rho \times T_2$ and apply Theorem 3.1 with $t \stackrel{\text{def}}{=} \dim T_2 = n - d$.

Let U be a connected subset of $U_k^{(\varepsilon)} \in \mathcal{U}^{(\varepsilon)}$. By [McM05, §7],the k-dimensional subspace $L \stackrel{\text{def}}{=} a^{-1}\mathbf{V}_{k\varepsilon}(ax)$ as well as the discrete subgroup $\Lambda \stackrel{\text{def}}{=} L \cap x$ are independent of the choice of $a \in U$. By definition of $U_k^{(\varepsilon)}$, for any $a \in U$, $a\Lambda$ contains k vectors $v_i = v_i(a), i = 1, \ldots, k$ which span aL and satisfy

$$\|v_i\| \in [r, (1+k\varepsilon)r], \quad \text{where} \quad r \stackrel{\text{def}}{=} \alpha(ax). \tag{3.3}$$

In order to verify hypothesis (i) of Theorem 3.1, we need to show that there is at least one j for which $U \cap M_j = \varnothing$. Since $\ker P_1 \cap \cdots \cap \ker P_d = \{0\}$ and $\dim L = k \geq 1$, it suffices to show that whenever $U \cap M_j \neq \varnothing$, $L \subset \ker P_j$. The face F_j of Δ_ρ consists of those elements $a_1 \in T_1$ which expand vectors in V_j by a factor of e^ρ. If $U \cap M_j \neq \varnothing$ then there is $a \in T_2, a_1 \in F_j$ so that $a_1 a \in U$. Now (3.2), (3.3) and the choice of η and C ensure that the vectors $v_i = v_i(a_1 a)$ satisfy $P_j(v_i) = 0$. Therefore $L \subset \ker P_j$.

It remains to verify hypothesis (ii) of Theorem 3.1. Let U be a connected subset of an intersection $U_{i_1} \cap \cdots \cap U_{i_k} \cap (\Delta_\rho \times T_2)$ and let $L_{i_j} \stackrel{\mathrm{def}}{=} a^{-1}\mathbf{V}_{i_j\varepsilon}(ax)$ and $\Lambda_{i_j} \stackrel{\mathrm{def}}{=} L_{i_j} \cap x$. As remarked above, L_{i_j}, Λ_{i_j} are independent of $a \in U$.

By the definition of the L_{i_j}'s we have that $L_{i_j} \subsetneq L_{i_{j+1}}$ and so they form a flag $\mathscr{F}$ as in (2.1). Lemma 2.5 applies and we deduce that

$$A_{\mathscr{F}} = \cap_{j=1}^k A_{L_{i_j}} \text{ is of co-dimension} \geq k \text{ in } A. \tag{3.4}$$

For each $a \in U$ and each j let $\left\{v_\ell^{(j)}(a)\right\} \in a\Lambda_{i_j}$ be the vectors spanning aL_{i_j} which satisfy (3.3). Let $u_\ell^{(j)}(a) \stackrel{\mathrm{def}}{=} a^{-1}v_\ell^{(j)} \in \Lambda_{i_j}$. Observe that:

(a) $\mathrm{span}_{\mathbb{Z}}\left\{u_\ell^{(j)}(a)\right\}$ is of finite index in Λ_{i_j} and in particular, $u_{i_1}^{(j)}(a) \wedge \cdots \wedge u_{i_j}^{(j)}(a)$ is an integer multiple of $\pm w_{\Lambda_{i_j}}$. As a consequence $\left\|a w_{\Lambda_{i_j}}\right\| \leq \left\|v_{i_1}^{(j)}(a) \wedge \cdots \wedge v_{i_j}^{(j)}(a)\right\|$.

(b) Because of (3.3) we have that $\left\|v_{i_1}^{(j)}(a) \wedge \cdots \wedge v_{i_j}^{(j)}(a)\right\| < C$ for some constant C depending on n alone.

It follows from (a),(b) and Lemma 2.4 that there exist $R > 0$ and an element $b \in T_2$ so that

$$U \subset \Delta_\rho \times \left\{a \in T_2 : \forall i_j, \left\|a w_{\Lambda_{i_j}}\right\| < C\right\} \subset T_1 \times \{a \in T_2 : \mathbf{d}(a, bA_{\mathscr{F}}) \leq R\}.$$

By (3.4) we deduce that if $p_2 : A \to T_2$ is the projection associated with the decomposition $A = T_1 \times T_2$ then $p_2(U)$ is $(R', s+t-k)$-almost affine, where R' depends only on R, ρ. This concludes the proof. $\square$

4. Proof of Theorem 3.1

In this section we will prove Theorem 3.1. Our proof gives an elementary alternative proof of McMullen's result. Moreover it shows that McMullen's hypothesis that the inradius of the cover is positive, is not essential.

Below X will denote a second countable locally connected metric space. We will use calligraphic letters like $\mathcal{U}$ for collections of sets.

The symbol $\mathrm{mesh}(\mathcal{A})$ will denote the supremum of the diameters of the sets in $\mathcal{A}$. The symbol $\mathrm{Leb}(\mathcal{A})$ will denote the Lebesgue number of a cover $\mathcal{A}$, i.e. the supremum of all numbers r such that each ball of radius r in X is contained in some element of $\mathcal{A}$. The symbol $\mathrm{ord}(\mathcal{A})$ will denote the largest number of distinct elements of $\mathcal{A}$ with non-empty intersection.

Definition 4.1. Let $\{X_j\}_{j\in\mathscr{J}}$ be a collection of subsets of X. We consider each X_j as an independent metric space with the metric inherited from X, and say that the collection is *uniformly of asymptotic dimension* $\leq n$ if for every $r > 0$ there is $R > 0$ such that for every $j \in \mathscr{J}$ there is an open cover $\mathcal{X}_j$ of X_j such that

- $\mathrm{mesh}(\mathcal{X}_j) \leq R$.
- $\mathrm{Leb}(\mathcal{X}_j) > r$.
- $\mathrm{ord}(\mathcal{X}_j) \leq n + 1$.

As an abbreviation we will sometimes write 'asdim' in place of 'asymptotic dimension'.

Recall that a cover of X is *locally finite* if every $x \in X$ has a neighborhood which intersects finitely many sets in the cover. We call the intersection of k distinct elements of $\mathcal{A}$ a *k-intersection,* and denote the union of all k-intersections by $[\mathcal{A}]^k$. We will need the following two Propositions for the proof of Theorem 3.1. We first prove Theorem 3.1 assuming them and then turn to their proof.

Proposition 4.2. Let $\mathcal{A}$ be a locally finite open cover of X such that $\mathrm{ord}(\mathcal{A}) \leq m$ and the collection of components of the k-intersections of $\mathcal{A}$, $1 \leq k \leq m$, is uniformly of asdim $\leq m - k$. Then $\mathcal{A}$ can be refined by a uniformly bounded open cover of order at most m.

Note that McMullen's theorem, namely the case $s = 0$ of Theorem 3.1, already follows from Proposition 4.2, since by a theorem of Lebesgue, a uniformly bounded open cover of $\mathbb{R}^t$ is of order at least $t + 1$.

Proposition 4.3. Let Δ_1 and Δ_2 be simplices, $X = \Delta_1 \times \Delta_2$, $p_i : X \to \Delta_i$ the projections and $\mathcal{A}$ a finite open cover of X such that for every $A \in \mathcal{A}$ and $i = 1, 2$ the set $p_i(A)$ does not meet at least one of the faces of Δ_i. Then $\mathrm{ord}(\mathcal{A}) \geq \dim \Delta_1 + \dim \Delta_2 + 1$.

Proof of Theorem 3.1. Let $m \stackrel{\mathrm{def}}{=} \dim M = s + t$, and suppose by contradiction that $\mathrm{ord}(\mathcal{U}) \leq m$. Since every cover of M has a locally finite refinement, there is no loss of generality in assuming that $\mathcal{U}$ is locally finite. Replacing $\mathcal{U}$ with the set of connected components of elements of $\mathcal{U}$, we may assume that all elements of $\mathcal{U}$ are connected. For any r_0, and any bounded set Y, the product space $Y \times \mathbb{R}^d$ can be covered by a cover of order $d+1$ and Lebesgue number greater than r_0. Hence our hypothesis (ii) implies that for each $k = 1, \ldots, m$, the collection of connected components of intersections of k distinct elements of $\mathcal{U}$ is uniformly of asymptotic dimension at most $m-k$. Therefore we can apply Proposition 4.2 to assume that $\mathcal{U}$ is uniformly bounded and of order at most m. Take a sufficiently large t-dimensional simplex $\Delta_1 \subset \mathbb{R}^t$ so that the projection of every set in $\mathcal{U}$ does not intersect at least one of the faces of Δ_1. We obtain a contradiction to Proposition 4.3. $\square$

For the proofs of Propositions 4.2, 4.3 we will need some auxiliary lemmas.

Lemma 4.4. *Let $\{G_i : i \in \mathscr{I}\}$ be a locally finite collection of open subsets of X, and let Z be an open subset such that for each $i \neq j$, $G_i \cap G_j \subset Z$. Then there are disjoint open subsets $\mathbf{E}_i, i \in \mathscr{I}$, such that for any i*

$$G_i \smallsetminus Z \subset \mathbf{E}_i \subset G_i.$$

Proof. Let $G = \bigcup_{i \in \mathscr{I}} G_i$. Without loss of generality we can assume that $X = G \cup Z$. Define $F_i \stackrel{\text{def}}{=} G_i \smallsetminus Z, F \stackrel{\text{def}}{=} G \smallsetminus Z$. Then the sets F_i are closed and disjoint, and since the collection $\{G_i\}$ is locally finite, the sets $F \smallsetminus F_i$ are closed as well. Denote by $\mathbf{d}$ the metric on X as well as the distance from a point to a closed subset. Then it is easy to verify that the sets

$$\mathbf{E}_i \stackrel{\text{def}}{=} \{x \in G_i : \mathbf{d}(x, F_i) < \mathbf{d}(x, F \smallsetminus F_i)\}$$

satisfy the requirements. $\square$

We denote the nerve of a cover $\mathcal{A}$ by $\text{Nerve}(\mathcal{A})$, and consider it with the metric topology induced by barycentric coordinates. Given a partitition of unity subordinate to a cover $\mathcal{A}$ of X, there is a standard construction of a map $X \to \text{Nerve}(\mathcal{A})$; such a map is called a *canonical map.*

Lemma 4.5. *Let a locally connected metric space Y be the union of two open subsets $\mathbf{D}$ and $\mathbf{E}$, and let $\mathcal{D}$ and $\mathcal{E}$ be open covers of $\mathbf{D}$ and $\mathbf{E}$ respectively, with bounded* mesh *and* ord*, and such that if $C \subset \mathbf{D} \cap \mathbf{E}$ is a connected subset contained in an element of $\mathcal{D}$, then it is contained in an element of $\mathcal{E}$. Then, there is an open cover $\mathcal{Y}$ of Y such that:*

(1) The cover $\mathcal{Y}$ refines $\mathcal{D} \cup \mathcal{E}$.
(2) $\text{mesh}(\mathcal{Y}) \leq \max\left(\text{mesh}(\mathcal{D}), \text{mesh}(\mathcal{E})\right)$.

(3) $\mathrm{ord}(\mathcal{Y}) \leq \max\left(\mathrm{ord}(\mathcal{D})+1, \mathrm{ord}\left(\mathcal{E}\right)\right)$.

Proof. Let $\mathrm{ord}(\mathcal{D}) = n+1$, let $\mathbf{A} \stackrel{\mathrm{def}}{=} \mathrm{Nerve}(\mathcal{D})$, and let $\pi : \mathbf{D} \to \mathbf{A}$ be a canonical map. Take an open cover of $\mathcal{A}$ of $\mathbf{A}$ such that $\mathrm{ord}(\mathcal{A}) \leq n+1$ and $\pi^{-1}(\mathcal{A})$ refines $\mathcal{D}$. Let $f : Y \to [0,1]$ be a continuous map such that $f|_{Y \smallsetminus \mathbf{E}} \equiv 0$ and $f|_{Y \smallsetminus \mathbf{D}} \equiv 1$. Set $\mathbf{C} \stackrel{\mathrm{def}}{=} f^{-1}\left([0,1)\right)$ and

$$g : \mathbf{C} \to \mathbf{B} \stackrel{\mathrm{def}}{=} \mathbf{A} \times [0,1], \quad g(c) \stackrel{\mathrm{def}}{=} (\pi(c), f(c)).$$

Since $\dim \mathbf{B} \leq n+1$ there is an open cover $\mathcal{B}$ of $\mathbf{B}$ such that $\mathrm{ord}(\mathcal{B}) \leq n+2$, the projection of $\mathcal{B}$ to $\mathbf{A}$ refines $\mathcal{A}$ and the projection of $\mathcal{B}$ to $[0,1]$ is of mesh $< 1/2$. Let $\mathcal{C}$ denote the collection of connected components of sets $\{g^{-1}(B) : B \in \mathcal{B}\}$. By construction $\mathcal{C}$ refines $\mathcal{D}$. Moreover $\mathrm{ord}(\mathcal{C}) \leq n+2$ and no element of $\mathcal{C}$ meets both $f^{-1}(0)$ and $f^{-1}\left(\left[\frac{1}{2},1\right]\right)$. Then for every $C \in \mathcal{C}$ which meets $f^{-1}\left(\left[\frac{1}{2},1\right]\right)$ we have that $C \subset \mathbf{D} \cap \mathbf{E}$ and there is an element $D \in \mathcal{D}$ such that $C \subset D$ and hence there is $E \in \mathcal{E}$ such that $C \subset E$. We choose one such E and say that E *marks* C.

We now modify elements of $\mathcal{E}$, defining

$$\tilde{E} \stackrel{\mathrm{def}}{=} \left(E \cap f^{-1}\left(\left(\frac{1}{2}, 1\right]\right)\right) \cup \bigcup_{E \text{ marks } C} C.$$

Finally define $\mathcal{Y}$ as the collection of modified elements of $\mathcal{E}$ and the elements of $\mathcal{C}$ which do not meet $f^{-1}\left(\left[\frac{1}{2},1\right]\right)$. It is easy to see that $\mathcal{Y}$ has the required properties. ☐

Lemma 4.6. *Let Y be a locally connected metric space and let $\mathbf{D}, \mathbf{E}_i, i \in \mathscr{I}$ be open subsets which cover Y. Assume that the $\mathbf{E}_i$'s are disjoint, connected, and are uniformly of* asdim $\leq \ell$*. Let $\mathcal{D}$ be an open cover of $\mathbf{D}$ which is of bounded mesh and* $\mathrm{ord}\,\mathcal{D} \leq \ell$*. Then Y has an open cover $\mathcal{Y}$ which refines the cover $\mathcal{D} \cup \{\mathbf{E}_i : i \in \mathscr{I}\}$, is of bounded mesh and* $\mathrm{ord}\,\mathcal{Y} \leq \ell+1$.

Proof. Using the assumption that $\mathbf{E}_i$ is uniformly of asdim $\leq \ell$ we find an open cover $\mathcal{E}_i$ of $\mathbf{E}_i$ which is of uniformly bounded mesh, such that $\mathrm{ord}\,\mathcal{E}_i \leq \ell+1$ and $\mathrm{Leb}\,\mathcal{E}_i > \mathrm{mesh}\,\mathcal{D}$. We assume that the sets in $\mathcal{E}_i$ are subsets of $\mathbf{E}_i$ and $\mathrm{Leb}(\mathcal{E}_i)$ is determined with respect to the metric of Y restriced to $\mathbf{E}_i$. Let

$$\mathbf{E} \stackrel{\mathrm{def}}{=} \bigcup_{i \in \mathscr{I}} \mathbf{E}_i \quad \text{and} \quad \mathcal{E} \stackrel{\mathrm{def}}{=} \bigcup_{i \in \mathscr{I}} \mathcal{E}_i.$$

Clearly it suffices to verify that the hypotheses of Lemma 4.5 are satisfied. Indeed, by assumption the cover $\mathcal{D}$ is of bounded mesh and order, and $\mathcal{E}$ is of bounded mesh because of the uniform bound on mesh$(\mathcal{E}_i)$. We also have that ord$\mathcal{E} \leq \ell + 1$ because of the bounds ord$\mathcal{E}_i \leq \ell + 1$ and the fact that the $\mathbf{E}_i$ are disjoint. For the last condition, let a connected subset $C \subset \mathbf{D} \cap \mathbf{E}$ which is contained in an element of $\mathcal{D}$ be given. By the connectedness and disjointness of the $\mathbf{E}_i$'s we conclude that there exists i with $C \subset \mathbf{E}_i$. Because Leb $\mathcal{E}_i >$ mesh$\mathcal{D}$ we deduce that since C is contained in an element of $\mathcal{D}$ it must be contained in an element of $\mathcal{E}_i$ and in turn, it must be contained in an element of $\mathcal{E}$. $\square$

Proof of Proposition 4.2. Proceeding inductively in reverse order, for $k = m, \ldots, 1$ we will construct a uniformly bounded open cover $\mathcal{A}^k$ of $[\mathcal{A}]^k$ such that ord$(\mathcal{A}^k) \leq m + 1 - k$ and $\mathcal{A}^k$ refines the restriction of $\mathcal{A}$ to $[\mathcal{A}]^k$. The construction is obvious for $k = m$. Namely, our hypothesis and Definition 4.1 with $n = m - k = 0$ mean that $[\mathcal{A}]^m$ has a cover of bounded mesh and order 1, that is, we can just set $\mathcal{A}^m$ to be the connected components of $[\mathcal{A}]^m$. Assume that the construction is completed for $k+1$ and proceed to k as follows. First notice that for two distinct k-intersections A and A' of $\mathcal{A}$ the complements $A \smallsetminus [\mathcal{A}]^{k+1}$ and $A' \smallsetminus [\mathcal{A}]^{k+1}$ are disjoint. By Lemma 4.4, we can cover $[\mathcal{A}]^k \smallsetminus [\mathcal{A}]^{k+1}$ by a collection $\{\mathbf{E}_i : i \in \mathscr{I}\}$ of disjoint connected open sets such that every $\mathbf{E}_i$ is contained in a k-intersection of $\mathcal{A}$. In particular, the collection $\{\mathbf{E}_i : i \in \mathscr{I}\}$ is uniformly of asdim $\leq m - k$. We can therefore apply Lemma 4.6 with the choices $Y = [\mathcal{A}]^k, \mathbf{D} = [\mathcal{A}]^{k+1}, \mathcal{D} = \mathcal{A}^{k+1}$, the collection $\{\mathbf{E}_i : i \in \mathscr{I}\}$, and $\ell = m - k$, and obtain an open cover $\mathcal{Y}$ of $[\mathcal{A}]^k$ of order $\leq m-k+1$ that refines $\mathcal{D} \cup \{\mathbf{E}_i : i \in \mathscr{I}\}$ and in particular, refines $\mathcal{A}|_Y$. This completes the inductive step.

$\square$

Proof of Proposition 4.3. For every $A \in \mathcal{A}$ choose a vertex v_i^A of Δ_i so that $p_i(A)$ does not intersect the face of Δ_i opposite to v_i^A. Let $Y \stackrel{\text{def}}{=} \text{Nerve}(\mathcal{A})$ and let $f : X \to X$ be the composition of a canonical map $X \to Y$ and a map $Y \to X$ which is linear on each simplex of Y and sends the vertex of Y related to $A \in \mathcal{A}$ to the point $(v_1^A, v_2^A) \in X$. Take a point $x \in \partial\Delta_1 \times \Delta_2$. Then $p_1(x)$ belongs to a face Δ_1' of Δ_1 and hence for every $A \in \mathcal{A}$ containing x we have that $v_1^A \in \Delta_1'$. Thus both x and $f(x)$ belong to $\Delta_1' \times \Delta_2$. Applying the same argument to $\Delta_1 \times \partial\Delta_2$ we get that the boundary ∂X is invariant under f and f restricted to ∂X is homotopic to the identity map of ∂X. If ord$(\mathcal{A}) \leq \dim \Delta_1 + \dim \Delta_2$ then $\dim Y \leq \dim X - 1$ and hence there is an interior point a of X not covered by $f(X)$. Take a retraction $r : X \smallsetminus \{a\} \to \partial X$. Then the identity map of ∂X factors up to homotopy through the contractible space X which contradicts the non-triviality of the reduced homology of ∂X. □

See §5 for another proof of Proposition 4.3.

5. Another argument for Theorem 3.1

In this section we will sketch another proof of Theorem 3.1. The proof, suggested by Roman Karasev, proceeds by reducing the theorem to its special case $s = 0$.

Let Δ_{CFK} denote the *Coxeter-Freudenthal-Kuhn simplex*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \le x_1 \le \cdots \le x_n \le 1\},$$

and let Γ be the group generated by isometric reflections of $\mathbb{R}^n$ in the facets of Δ_{CFK}. Then it is known [Cox34] that Γ acts discretely on $\mathbb{R}^n$ with fundamental domain Δ_{CFK} (Γ is the so-called *affine Coxeter group of type* $\tilde{A}_n$). Using this fact, we prove Theorem 3.1 as follows.

Recall that the case $s = 0$ of the Theorem was proved by McMullen, see [McM05, Thm. 5.1]. According to this result, a cover $\mathcal{V}$ of $\mathbb{R}^m$ with $\mathrm{Leb}(\mathcal{V}) > 0$ has order at least $m+1$, provided it satisfies the following analogue of (ii):

(ii)' There is R so that for connected component V of the intersection of $k \le m$ distinct elements of $\mathcal{V}$ is $(R, m-k)$-almost affine.

We remark that McMullen assumed that $\mathcal{V}$ has positive inradius, i.e. there is $r > 0$ such that for any $x \in \mathbb{R}^m$, there is an element of $\mathcal{V}$ containing the ball of radius r around x. However as we remarked above, this hypothesis is not essential.

Setting $m \stackrel{\mathrm{def}}{=} s + t$, starting with a cover of M satisfying (i) and (ii) we will form a cover of $\mathbb{R}^m$ satisfying (ii)'.

Clearly there is no loss of generality in assuming that $\Delta = \Delta_{CFK}$. Let $\varphi : \mathbb{R}^m \to \Delta \times \mathbb{R}^t$ be the map which sends (x, y), where $x \in \mathbb{R}^s, y \in \mathbb{R}^t$ to (x', y) where x' is the representative of the orbit Γx in Δ. Let $\mathcal{V}$ be the cover of $\mathbb{R}^{s+t}$ obtained by pulling back the cover $\mathcal{U}$. For each j, let x_j be the vertex of Δ opposite F_j and let Γ_j be the finite subgroup of Γ fixing x_j. Then $\Gamma_j \Delta$ is a polytope all of whose boundary faces are images of F_j under Γ_j. In light of assumption (i), this implies that any connected component of any $V \in \mathcal{V}$ is within a uniformly bounded distance of $\{v\} \times \mathbb{R}^t$ for some $v \in V$. Therefore (ii)' holds for $\mathcal{V}$, for $k \le s$, while for $k > s$, (ii)' for $\mathcal{V}$ is implied by (ii) for $\mathcal{U}$. By McMullen's theorem, the order of $\mathcal{V}$ is at least $s + t + 1$, and therefore the same holds for $\mathcal{U}$.

A similar argument, also suggested by Roman Karasev, gives another proof of Proposition 4.3. Namely suppose that for $i = 1, 2$, $\Delta_i \subset \mathbb{R}^{n_i}$ is realized concretely as the Coxeter-Freudenthal-Kuhn simplex of dimension n_i. Let $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_i is the group generated by reflections in the facets of Δ_i. Then a cover of $\Delta_1 \times \Delta_2$ gives rise to a cover of $\mathbb{R}^{n_1+n_2}$ by open sets of uniformly bounded diameter, and hence Lebesgue's theorem implies that there is a point which is covered $n_1 + n_2 + 1$ times.

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Dept. of Mathematics, Ben Gurion University, Be'er Sheva, Israel mlevine@math.bgu.ac.il

Dept. of Mathematics, Technion, Haifa, Israel ushapira@tx.technion.ac.il

Dept. of Mathematics, Tel Aviv University, Tel Aviv, Israel barakw@post.tau.ac.il