FULL ESCAPE OF MASS FOR THE DIAGONAL GROUP

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ABSTRACT. Building on the work of Cassels we prove the existence of infinite families of compact orbits of the diagonal group in the space of lattices which accumulate only on the divergent orbit of the standard lattice. As a consequence, we prove the existence of full escape of mass for sequences of such orbits, sharpening known results about possible partial escape of mass. The topological complexity of the visits of these orbits to a given compact set in the space of lattices is analyzed. Along the way we develop new tools to compute regulators of orders in a number fields.

1. INTRODUCTION

1.1. **Preliminary description of results.** Let $d \geq 2$ be an integer and let $X \overset{\text{def}}{=} \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ denote the space of unimodular lattices in $\mathbb{R}^d$. Let $A < \mathrm{SL}_d(\mathbb{R})$ be the group of diagonal matrices with positive diagonal entries. The aim of this paper is to establish the following result.

Theorem 1.1. *There exists a sequence of distinct compact A-orbits Ax_k such that any limit point of the form $x = \lim a_k x_k$, with $a_k \in A$, satisfies $x \in A\mathbb{Z}^d$.*

We note that in dimension $d = 2$ Peter Sarnak points out in [Sar07] a construction yielding the above result. In higher dimensions the analysis is more intricate.

Given a compact orbit $Ax \subset X$, let us denote by μ_{Ax} the unique A-invariant probability measure supported on it. Theorem 1.1 implies the following result which sharpens [ELMV09, Theorem 1.10] in which only partial escape of mass is claimed.

Corollary 1.2. *Let Ax_k be a sequence of compact orbits satisfying the conclusion of Theorem 1.1. Then the probability measures μ_{Ax_k} converge to the zero measure.*

Proof. If μ is an accumulation point of μ_{Ax_k} then by Theorem 1.1, we conclude that μ is supported on the divergent orbit of the standard lattice. By Poincaré recurrence theorem, the only finite A-invariant measure supported on $A\mathbb{Z}^d$ is the zero measure. $\square$

Our results are much more precise than Theorem 1.1 but as a consequence, more lengthy to state and depend on various notation to be presented later. We roughly describe their features. For $\epsilon > 0$ we set

$$X^{\geq \epsilon} \stackrel{\text{def}}{=} \{x \in X : \ell(x) \geq \epsilon\} \text{ where } \ell(x) \stackrel{\text{def}}{=} \min\{||v|| : 0 \neq v \in x\}. \tag{1.1}$$

In §4 we isolate a family of compact orbits $\mathcal{F}$, (which in §5 is shown to be infinite), for which there exists a function $\eta : \mathcal{F} \to (0, \infty)$ with $\eta(Ax) \to 0$ as $Ax \in \mathcal{F}$ varies, such that for any sufficiently small $\delta > 0$ and for all but finitely many $Ax \in \mathcal{F}$, the following statements are satisfied (see Propositions 4.5, 4.8, and Theorem 4.14 for precise statements):

(1) $\mu_{Ax}(X^{\geq \delta}) \leq \eta(Ax)$.
(2) $\forall y \in Ax \cap X^{\geq \delta}$, $\mathrm{d}(y, A\mathbb{Z}^d) \leq \eta(Ax)$.
(3) The number of connected components of $Ax \cap X^{\geq \delta}$ is at least $(d-1)!$.

In fact, the results are sharper in the sense that $\eta(Ax)$ is explicit and more interestingly, δ could be chosen to be an explicit function of the orbit such that $\delta(Ax) \to 0$ as $Ax \in \mathcal{F}$ varies.

From the point of view of algebraic number theory, Theorem 4.14, which gives new tools for showing that certain collections of units in an order $\mathcal{O}$ generate the group of units $\mathcal{O}^\times$ up to torsion[1], might be of interest on its own. This result is (on the face of it) unrelated to the discussion on escape of mass but its proof uses the analysis yielding Theorem 1.1 in a fundamental way. For instance, given a vector of distinct integers $\mathbf{m} \in \mathbb{Z}^d$, let θ_k be a root of the polynomial $p_k(x) \stackrel{\text{def}}{=} \prod_1^d (x - km_i) - 1$. Then, it follows from Theorem 4.14 and the analysis in §5, that for all large enough k, the collection $\{\theta_k - km_i\}_{i=1}^d$ generates the group of units of the order $\mathbb{Z}[\theta_k]$ up to roots of unity.

1.2. **Cassels' work on Minkowski's conjecture.** This paper originated from a simple observation made while reading Cassels' paper [Cas52] showing that the value 2^{-d} is not isolated in the Minkowski spectrum.

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[1]This result is only applicable under certain assumptions relating the discriminant of the order to the geometry of the collection of units.

Given a lattice $x \in X$ let us define

$$\mu(x) = \sup_{\mathbf{v} \in \mathbb{R}^d} \inf \left\{ \prod_1^d |w_i - v_i| : \mathbf{w} \in x \right\},$$

and let us define the the *Minkowski spectrum* in dimension d to be $\mathscr{M} \overset{\text{def}}{=} \{\mu(x) : x \in X\}$. A famous conjecture attributed to Minkowski asserts that $\mathscr{M} \subset [0, 2^{-d}]$ and that $\mu(x) = 2^{-d}$ if and only if $x \in A\mathbb{Z}^d$.

In [Cas52] Cassels gives a construction of a sequence $\{x_k\}$ of lattices having compact A-orbits with the property that $\lim_k \mu(x_k) = 2^{-d}$. Taking into account the fact that μ is A-invariant and upper semi-continuous (in the sense that if $x_k \to x$ then $\limsup \mu(x_k) \leq \mu(x)$), one arrives at the inevitable conclusion that, assuming Minkowski's conjecture holds in dimension d, the sequence x_k from Cassels' construction must satisfy the conclusion of Theorem 1.1; thus proving it in any dimension in which Minkowski's conjecture is known to hold. To this date Minkowski's conjecture has been validated up to dimension $n = 9$ (see for example [McM05],[HGRS11],[KHGR15],[SW] for recent accounts).

As it turns out, in order to prove Theorem 1.1 one does not need Minkowski's conjecture as an input and it could be derived by a careful analysis of Cassels' construction which we slightly generalize in §5.

1.3. **Homogeneous dynamics context.** Another context for interpreting the results is the comparison between unipotent dynamics and (higher-rank) diagonalizable dynamics. See [ELMV09] for a thorough discussion explaining the fundamental differences between these two worlds. In this spirit, we remark that Theorem 1.1 and its corollary are in sharp contrast to the rigidity exhibited by sequences of periodic orbits Hx_k if one assumes $H < \mathrm{SL}_d(\mathbb{R})$ is generated by unipotent elements. It is a consequence of a result of Mozes and Shah [MS95] (relying on results by Dani and Margulis), that if $Hx_k \subset X$ is such a sequence and there is a fixed compact set $K \subset X$ such that $Hx_k \cap K \neq \varnothing$, then any accumulation point of the sequence μ_{Hx_k} is a probability measure[2]. We note that there exists a compact set K such that $Ax \cap K \neq \varnothing$ for any $x \in X$ (see Theorem 4.1 and the discussion following it for references).

Having this comparison in mind, we find the following open questions (for $d \geq 3$) natural and interesting.

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[2]The information on such accumulation points is much more informative but we state here only the part relevant for us.

(Q1) Does there exists a sequence of compact orbits Ax_k such that μ_{Ax_k} converges to a non-ergodic measure?

(Q2) Does there exists a sequence of compact orbits Ax_k such that μ_{Ax_k} exhibits strictly partial escape of mass (i.e. which converges to a non-zero measure of total mass < 1)?

(Q3) Does there exists a sequence of compact orbits Ax_k such that μ_{Ax_k} converges to a limit whose ergodic decomposition contains a periodic A-invariant measure with positive weight?

Another interesting question is whether or not one can construct a sequence of compact orbits Ax_k with μ_{Ax_k} exhibiting full (or even partial) escape of mass such that the lattices x_k are geometric embeddings of full modules in a fixed number field. We note though that in dimension 2 for example, due to the results in [AS], the arithmetic relations between the full modules in such an example must involve infinitely many primes. Interestingly, in positive characteristics this situation breaks and there are examples in dimension 2 of sequences of compact orbits arising from a fixed quadratic field and sharing arithmetic relation involving a single prime which produce full escape of mass. For details see [KPS].

2. INTUITION

As often happens, the idea behind the proofs is simple but it is not unlikely that without clarifications it may appear quite hidden behind the details. We therefore try to explain it briefly and along the way present some of the notation that we shall use. The diagonal group is naturally identified with the hyperplane

$$\mathbb{R}_0^d \overset{\text{def}}{=} \left\{ \mathbf{t} \in \mathbb{R}^d : \sum_1^d t_i = 0 \right\}.$$

As the dimension of this hyperplane will appear frequently in our formulas, we henceforth denote

$$n \overset{\text{def}}{=} \dim \mathbb{R}_0^d = d - 1.$$

Given a compact orbit Ax, it corresponds in a natural way[3] to a lattice $\Delta < \mathbb{R}_0^d$ and on choosing a fundamental domain F (say a parallelepiped) for this lattice, we may identify the orbit with F via the map $\mathbf{t} \mapsto a(\mathbf{t})x$, where

$$a(\mathbf{t}) \stackrel{\text{def}}{=} \operatorname{diag}\left(e^{t_1}, \ldots, e^{t_d}\right). \tag{2.1}$$

We think of $\mathbf{t} \in \mathbb{R}_0^d$ as the 'time parameter', and as it varies, the point on the orbit corresponding to it varies as well. If it so happens that x contains a very short vector, say of length ℓ, at time 0 (and hence at any time from Δ), then one can estimate quite easily a radius $r = r(\ell)$ such that for any time in a ball of radius r around a corner of F the vector is still short. A strategy for deducing that the orbit spends most of the time near infinity is to establish that the volume of these neighbourhoods occupies most of the volume of F. The success of this strategy depends therefore first and foremost on the relationship between the volume of F and the length ℓ but also on the 'geometry' of F as is illustrated in Figure 1 below.
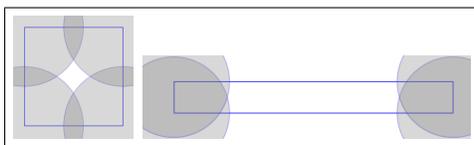

FIGURE 1. The square on the left is a tamed fundamental domain of area 1 and the grey circles cover most of its area. The rectangle on the right is a skewed fundamental domain and the circles fail to cover a significant proportion of it. The grey areas represents times in which the orbit is near infinity.

This strategy suffices to establish partial escape of mass in certain examples but in order to obtain the sharper results of this paper, and in particular, full escape of mass, one needs to modify it slightly.

The first modification has to do with the norm. In the above description we mentioned an estimate of the radius of a euclidean ball around points of Δ in which we have control on the length of some short vector. It turns out that this estimate becomes sharper when we replace the euclidean norm with a slight modification of it which takes advantage of the fact that $\mathbb{R}_0^d$ sits in $\mathbb{R}^d$. Given $\mathbf{t} \in \mathbb{R}_0^d$, we denote

$$\lceil \mathbf{t} \rceil = \max\left\{t_i : 1 \leq i \leq d\right\}.$$

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[3]For the sake of the current discussion the reader might envision Δ as the image under the logarithm of the stabilizer or x but in fact, the lattice we consider could potentially be slightly larger. See §4 for the definition.

Note that there exists a constant $c = c(d) > 0$ such that for $\mathbf{t} \in \mathbb{R}_0^d$, $c^{-1}||\mathbf{t}|| \leq \lceil \mathbf{t} \rceil \leq c||\mathbf{t}||$. The ball of radius r is then replaced by the set $\left\{ \mathbf{t} \in \mathbb{R}_0^d : \lceil \mathbf{t} \rceil \leq r \right\}$ which as the reader can easily convince himself, is a simplex. Another modification to the above strategy, and which is tightly related to the switch from euclidean balls to simplexes, is as follows. Since we have good estimates for the length of vectors when the time parameter is confined to a simplex, it will be more convenient not to work with a fundamental domain given by the parallelepiped obtained from a basis of Δ, but with a certain simplex which one can cook up from this basis. This brings us to a situation where the balls and the rectangles in Figure 1 are replaced respectively by simplexes centered around points of Δ and a simplex of the same type which contains a fundamental domain as illustrated in Figure 2 below.
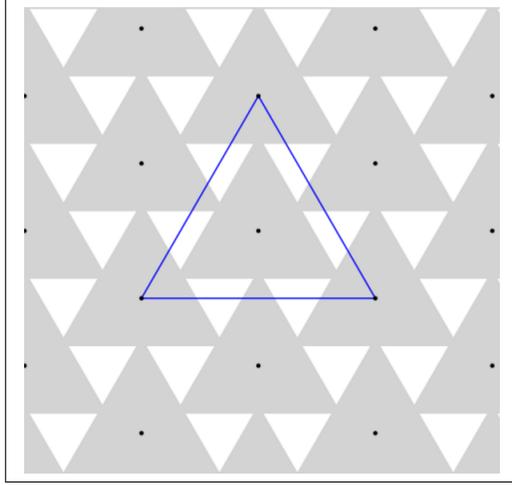

FIGURE 2. The black dots are the points of Δ, the grey simplexes are the areas where we have control on the short vector, and the indicated blue triangle border a simplex which contains a fundamental domain for Δ.

The structure of the paper is as follows. In §3 we introduce the concept of a *simplex set* Φ of a lattice $\Delta < \mathbb{R}_0^d$ and various quantities attached to it. This section culminates in Proposition 3.8 which is fundamental for the analysis. In a nutshell, it gives the appropriate mathematical content to the illustration in Figure 2.

In §4 we study lattices with compact orbits of the diagonal group possessing nice simplex sets (which we refer to as M-tight simplex sets). Having an M-tight simplex set allows us to translate Proposition 3.8 and establish the main results which are Propositions 4.5, 4.8, and

Theorem 4.14. In particular, for these lattices we will control the white area inside the blue simplex in Figure 2 and show that it is negligible compared to the covolume of Δ (see Proposition 4.5(3)). In §5 we give a construction and carefully analyze it, showing that the results of §4 may be applied. This construction is a slight generalization of the one given by Cassels.

3. SIMPLEX SETS AND A COVERING RADIUS CALCULATION

3.1.   We begin by fixing some notation and terminology. For concreteness we fix the supremum norm $||\mathbf{t}|| \overset{\text{def}}{=} \max_{1 \leq i \leq d} |t_i|$ on $\mathbb{R}^d$ and work only with this norm and the operator norm corresponding to it. In particular, we obtain this way a norm on $\mathrm{GL}(\mathbb{R}^d_0)$. By $A \ll B$ we mean that there exists a constant c such that $A \leq cB$. If the constant depends on some parameter M, we indicate this by writing $\ll_M$. An exception to this rule is the dependency of the constant in the dimension which we do not record in the notation (thus if not indicated otherwise, implicit constants are universal but may depend on d). Similar conventions are used with big O and small o notation. We write $A \asymp B$ to indicate that $A \ll B$ and $B \ll A$. For example, as noted in §2,

$$\forall \mathbf{t} \in \mathbb{R}^d_0, \ \ ||\mathbf{t}|| \asymp \lceil \mathbf{t} \rceil. \tag{3.1}$$

Definition 3.1. A *simplex set* $\Phi \subset \mathbb{R}^d_0$ is a spanning set of cardinality d such that $\sum_{\mathbf{t} \in \Phi} \mathbf{t} = 0$. The *associated simplex* is defined as the convex hull of Φ; $S_\Phi \overset{\text{def}}{=} \mathrm{conv}(\Phi)$, and the *associated lattice* is $\Delta_\Phi \overset{\text{def}}{=} \mathrm{span}_{\mathbb{Z}} \Phi$.

Although the concept of a simplex set is almost equivalent to a choice of a basis for the associated lattice, it will be more convenient to work with it.

Similar to the standard lattice and standard basis in euclidean space it will be convenient to introduce a certain "standard" simplex set: Let

$$\Phi_* \overset{\text{def}}{=} \{\mathbf{b}_j\}_{j=1}^d, \ \text{where,} \ \mathbf{b}_j \overset{\text{def}}{=} (1, \ldots, \underbrace{-n}_{j\text{'th place}}, \ldots, 1)^t,$$

and denote $\Delta_* \overset{\text{def}}{=} \Delta_{\Phi_*}$. As will be shown in the proof of Lemma 3.7, Δ_* is a dilated copy of the orthogonal projection of $\mathbb{Z}^d$ to $\mathbb{R}^d_0$.

Definition 3.2. Given a simplex set Φ, we define its *distortion map* to be an element $h_\Phi \in \mathrm{GL}(\mathbb{R}^d_0)$ satisfying $h_\Phi \Phi_* = |\Delta_\Phi|^{-\frac{1}{n}} \Phi$ (and as a consequence $h_\Phi \Delta_* = |\Delta_\Phi|^{-\frac{1}{n}} \Delta_\Phi$). Here $|\Delta_\Phi|$ is the covolume of Δ_Φ in $\mathbb{R}^d_0$. We define the *distortion* of Φ to be $||h_\Phi||$.

Note that there is no canonical choice of h_Φ but the $d!$ possible choices differ by precomposing with a permutation matrix and thus the distortion $||h_\Phi||$ is well defined and this ambiguity should not cause any harm.

Most of the assertions regarding a general simplex set Φ will be proved by establishing them first to Φ_* and then applying the linear map $|\Delta_\Phi|^{\frac{1}{n}} h_\Phi$ that maps Φ_* to Φ.

A related quantity to the distortion is the following.

Definition 3.3. Given a simplex set Φ we set

$$\xi_\Phi \stackrel{\text{def}}{=} \max_{\mathbf{t} \in \Phi} \lceil \mathbf{t} \rceil.$$

We collect a few elementary facts about the concepts introduced above.

Lemma 3.4.   (1) For any simplex set Φ, the distortion of Φ satisfies $||h_\Phi|| \asymp |\Delta_\Phi|^{-\frac{1}{n}} \xi_\Phi$.
     (2) If h_Φ is a distortion map with $||h_\Phi|| \leq T$ then for any $\mathbf{t} \in \mathbb{R}_0^d$, $||\mathbf{t}|| \ll_T ||h_\Phi \mathbf{t}||$.
     (3) If h_Φ is a distortion map with $||h_\Phi|| \leq T$, then $\xi_\Phi \ll_T \min\{||\mathbf{t}|| : \mathbf{t} \in \Phi\}$.

Proof. (1). It is straightforward that $|\Delta_\Phi|^{\frac{1}{n}} ||h_\Phi|| \asymp \max\{||\mathbf{t}|| : \mathbf{t} \in \Phi\}$ and so the results follows from the definition of ξ_Φ and (3.1).

(2) Because the determinant of a distortion map h_Φ is a constant, (independent of Φ), we have that $||h_\Phi^{-1}|| \ll ||h_\Phi||^n$ and the claim follows.

(3). This follows from (1), (2).

$\square$

Lemma 3.5. *Let Φ be a simplex set. Then, $|\Delta_\Phi|^{\frac{1}{n}} \ll \xi_\Phi$.*

Proof. For each $\mathbf{t} \in \Phi$ we have that $||\mathbf{t}|| \ll \lceil \mathbf{t} \rceil \leq \xi_\Phi$ so that Δ_Φ is spanned by a basis of vectors whose lengths are $\ll \xi_\Phi$. It follows that $|\Delta_\Phi| \ll \xi_\Phi^n$. Extracting the n'th root we obtain the desired claim. $\square$

3.2.   We proceed by introducing a collection of points $\mathbf{W}_\Phi$ associated to a simplex set Φ which will play an important role in our considerations. These points are indicated in red on the boundary of the blue triangle in Figure 3 below, which represents $\frac{n}{2} S_\Phi$. The black points represent Δ_Φ and the grey simplexes represent $\Delta_\Phi + (1 - \rho)\frac{n}{2} S_\Phi$ for some $0 \leq \rho \leq 1$.

Let $\mathscr{S}$ denote the group of permutation matrices in $\mathrm{GL}_d(\mathbb{R})$ and let $\theta \in \mathscr{S}$ denote the standard cycle permutation; that is, $\theta \mathbf{e}_i = \mathbf{e}_{i+1 \bmod d}$, where $\{\mathbf{e}_i\}_{i=1}^d$ denotes the standard basis for $\mathbb{R}^d$.
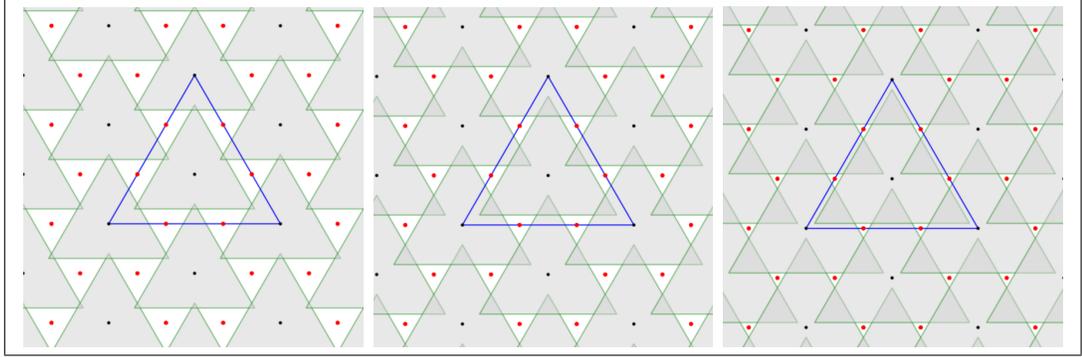
FIGURE 3. Envision the grey triangles being dilated simultanuously around their centers. The red (bigger) points are the critical points which are covered last (so called *deep holes*). See Proposition 3.8.

Definition 3.6. For any $\tau \in \mathscr{S}$ let

$$\mathbf{w}_\tau^{\Phi_*} \overset{\text{def}}{=} \tau(d^{-1}\sum_{\ell=1}^d (\ell-1)\mathbf{b}_\ell)$$

and in turn, for a general simplex set Φ with distortion map h_Φ, let

$$\mathbf{w}_\tau^{\Phi} \overset{\text{def}}{=} |\Delta_\Phi|^{\frac{1}{n}} h_\Phi \mathbf{w}_\tau^{\Phi_*}. \tag{3.2}$$

Note that the indexation in (3.2) depends on the choice of the distortion map but this should cause no confusion. We further let $\mathbf{W}_\Phi \overset{\text{def}}{=} \{\mathbf{w}_\tau^{\Phi}\}_{\tau \in \mathscr{S}}$ and choosing a set $\mathscr{S}' \subset \mathscr{S}$ of representatives of the quotient $\mathscr{S}/\langle\theta\rangle$ we set[4] $\mathbf{W}_\Phi' \overset{\text{def}}{=} \{\mathbf{w}_\tau^{\Phi}\}_{\tau \in \mathscr{S}'}$.

When Φ is understood we usually write $\mathbf{w}_\tau$. We remark that given a simplex set Φ, the collection $\mathbf{W}_\Phi$ lies on the boundary of the dilation $\frac{n}{2}S_\Phi$ because $\frac{2}{n}\mathbf{w}_\tau$ is a convex combination of the elements in Φ with one of the coefficients being 0.

The following lemma analyzes the reduction of $\mathbf{W}_\Phi$ modulo Δ_Φ.

Lemma 3.7. *Let Φ be a simplex set.*

(1) For $\tau, \sigma \in \mathscr{S}$, $\mathbf{w}_\tau + \Delta_\Phi = \mathbf{w}_\sigma + \Delta_\Phi$ if and only if $\tau^{-1}\sigma \in \langle\theta\rangle$.

(2) If $\tau^{-1}\sigma \notin \langle\theta\rangle$ then there exists $\tau' \in \mathscr{S}$ such that for some $1 \le k \le n$, $\mathbf{w}_{\tau'} + k(\mathbf{w}_\tau - \mathbf{w}_\sigma) \notin \mathbf{W}_\Phi + \Delta_\Phi$. Moreover, if d is prime then any τ' satisfies this.

Statement (1) is illustrated in Figure 3 where one can see that the six red points on the boundary of the blue simplex decompose into two

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[4]The collections $\mathscr{S}', \mathbf{W}_\Phi'$ are not canonically defined.

classes modulo the black points. Statement (2) will be used in the proof of Theorem 4.14 to show that a certain lattice that a priori contains Δ_Φ is actually equal to it. This will allow us to prove in §5 that certain collections of units in certain orders generate the corresponding group of units modulo torsion.

Proof. It suffices to prove the lemma for the standard simplex set Φ_*. Let us denote by $p : \mathbb{R}^d \to \mathbb{R}^d_0$ the orthogonal projection. Note that if $\{\mathbf{e}_\ell\}$ denotes the standard basis for $\mathbb{R}^d$ then for any ℓ and for any $\sigma \in \mathscr{S}$

$$
\begin{aligned}
\mathbf{b}_\ell &= p(-d\mathbf{e}_\ell), \\
\Delta_* &= p(d\mathbb{Z}^d), \\
\mathbf{w}_\sigma &= p((\sigma(1,\ldots,d))^t).
\end{aligned}
\tag{3.3}
$$

We first show that for any τ, $\mathbf{w}_\tau - \mathbf{w}_{\tau\circ\theta^j} \in \Delta_*$. By the above, this is equivalent to showing that the difference $\tau(1,\ldots,d) - \tau\circ\theta^j(1,\ldots,d) \in d\mathbb{Z}^d + \ker p$. Clearly, it is enough to deal with the case $j = 1$ and moreover, post-composing with τ^{-1} we may assume $\tau = \text{id}$ (here we are using the fact that both $\mathbb{Z}^d$ and $\ker p$ are $\mathscr{S}$-invariant). Indeed,

$$
\begin{pmatrix} \frac{1}{2} \\ \vdots \\ d \end{pmatrix} - \theta \begin{pmatrix} \frac{1}{2} \\ \vdots \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} d \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \ker p + d\mathbb{Z}^d.
$$

We prove the second part of (1). Let $\tau, \sigma \in \mathscr{S}$ be such that $\tau^{-1}\sigma \notin \langle\theta\rangle$. We show that $\mathbf{w}_\tau - \mathbf{w}_\sigma \notin \Delta_*$, which is equivalent to $\tau(1,\ldots,d) - \sigma(1,\ldots,d) \notin d\mathbb{Z}^d + \ker p$. Post-composing with τ^{-1} and rewriting $\tau^{-1}\sigma$ as σ we are reduced to the case $\tau = \text{id}$. Moreover, using the already established first part of (1), we may replace σ with $\sigma\circ\theta^j$ and we choose j so that $\sigma\circ\theta^j$ fixes the d'th coordinate. To this end, assume by way of contradiction the existence of a non-trivial permutation σ of $1,\ldots,n$ such that the difference $(1,\ldots,n,d) - (\sigma(1),\ldots,\sigma(n),d)$ belongs to $d\mathbb{Z}^d + \ker p$. Alternatively, all the differences between the coordinates of $(1-\sigma(1),\ldots,n-\sigma(n),0)$ are equal modulo d. As the last coordinate is 0 this is equivalent to saying that $(1,\ldots,n) = (\sigma(1),\ldots,\sigma(n))$ modulo d. But since this equality involves only numbers between 1 to n it is equivalent to σ being trivial which contradicts our assumption and concludes the proof of (1).

The proof of (2) follows similar lines. First observe that because $\ker p \cap \mathbb{Z}^d = \{k\mathbf{1} : k \in \mathbb{Z}\}$, where $\mathbf{1} \overset{\text{def}}{=} (1,\ldots,1)^t$, it follows from (3.3) that $\{\mathbf{v} \in \mathbb{Z}^d : p(\mathbf{v}) \in \mathbf{W}_{\Phi_*} + \Delta_*\}$ can be characterized as the set of integer vectors having all residues modulo d as coordinates.

Similarly to the consideration above we reduce to the case where $\tau = \mathrm{id}$ and σ is non-trivial and fixes the d'th axis. Thus we need to find τ' and k such that $\tau'(1, \ldots, d) + k(1 - \sigma(1), \ldots, n - \sigma(n), 0)$ does not contain all the residues modulo d as coordinates. Since σ is non-trivial we know that for some $1 \leq j \leq n$, $j - \sigma(j) \neq 0$. If we choose τ' so that τ' fixes the d axis and also, $\tau'(j) = -(j - \sigma(j))$ modulo d then we see that for $k = 1$ the above vector will contain the residue 0 twice and therefore will not contain all residues. If d is assumed to be prime, then for any τ' we could solve the equation $\tau'(d) = \tau'(j) + k(j - \sigma(j))$ modulo d with some $1 \leq k \leq d$, and then the above vector will contain the residue $\tau'(d)$ twice yielding the same contradiction.

We note that when d is not prime it is possible to find τ', τ, σ such that $\tau^{-1}\sigma \notin \mathscr{S}$ but $\mathbf{w}_{\tau'} + k(\mathbf{w}_\tau - \mathbf{w}_\sigma) \in \mathbf{W}_{\Phi_*} + \Delta_*$ for all k. For example, when $d = 4$ one can take $\mathbf{w}_{\tau'} = \mathbf{w}_\tau = (3, 2, 1, 4), \mathbf{w}_\sigma = (1, 2, 3, 4)$. □

We now state a covering result which plays a crucial role in the analysis below and shows the significance of the dilation $\frac{n}{2}S_\Phi$ and the collection $\mathbf{W}_\Phi$ (see Figure 3). This is roughly a restatement of [Cas52, Lemma 1] but we give the full proof for completeness.

Proposition 3.8. *Let Φ be a simplex.*
  (1) $\mathbb{R}_0^d = \frac{n}{2}S_\Phi + \Delta_\Phi$.
  (2) $\mathbb{R}_0^d \smallsetminus (\frac{n}{2}S_\Phi^\circ + \Delta_\Phi) \subset \mathbf{W}_\Phi + \Delta_\Phi = \mathbf{W}'_\Phi + \Delta_\Phi$, where S° denotes the interior of S.
  (3) There exists a universal constant $c > 0$ such that for any $\rho \in (0, 1)$, $\mathbb{R}_0^d \smallsetminus ((1 - \rho)\frac{n}{2}S_\Phi + \Delta_\Phi) \subset \mathbf{W}'_\Phi + B_{c\rho\xi_\Phi} + \Delta_\Phi$.

As explained in Remark 3.9, the inclusion in (2) is in fact equality. This shows that the covering radius of Δ_Φ by S_Φ is exactly $\frac{n}{2}$.

Proof. We prove the proposition for the standard simplex set Φ_*. In this case, it follows from the definition (or from (3.3)), that $\mathbf{w}_\tau = \tau(\mathbf{w})$, where $\mathbf{w} = (n/2, n/2 - 1, \ldots, -n/2)$. The statement for general simplex sets follows by applying the linear map $|\Delta_\Phi|^{\frac{1}{n}}h_\Phi$ which takes Φ_* to Φ. Indeed, it is clear that the statements (1), (2), follow immediately and (3) follows from the fact that the above linear transformation has norm $\ll \xi_\Phi$ by Lemma 3.4(1).

We prove (1) and (2) for Φ_*. One of the benefits of working in the hyperplane $\mathbb{R}_0^d$ is the following observation:

$$\frac{n}{2}S_{\Phi_*} = \left\{ \mathbf{v} \in \mathbb{R}_0^d : \lceil \mathbf{v} \rceil \leq \frac{n}{2} \right\}. \tag{3.4}$$

Given $\mathbf{u} \in \mathbb{R}_0^d$, if $\lceil \mathbf{u} \rceil \geq \frac{n}{2}$ we describe a procedure by which
  (i) either $\mathbf{u} = \mathbf{w}_\tau$ for some $\tau \in \mathscr{S}$,

(ii) or we find $\mathbf{v} \in \Delta_*$ such that $\lceil \mathbf{u} + \mathbf{v} \rceil < \lceil \mathbf{u} \rceil$.

As long as (ii) occurs, we replace $\mathbf{u}$ by $\mathbf{u} + \mathbf{v}$ and iterate. As the set $\Delta_* + \mathbf{u}$ is discrete and the sets $\left\{ \mathbf{v} \in \mathbb{R}_0^d : \lceil \mathbf{v} \rceil \leq t \right\}$ are compact, we cannot iterate this procedure indefinitely, which means that after finitely many iterations we will arrive at a vector $\mathbf{u}' \in \mathbf{u} + \Delta_*$ with $\mathbf{u}' \in \frac{n}{2} S_{\Phi_*}^\circ$, or to one of the points $\mathbf{w}_\tau$. This will conclude the proof of (1) and (2) (the equality $\mathbf{W}_{\Phi_*} + \Delta_* = \mathbf{W}'_{\Phi_*} + \Delta_*$ follows from Lemma 3.7(1)).

The procedure is as follows: Let $\mathbf{u} \in \mathbb{R}_0^d$ be such that $\lceil \mathbf{u} \rceil \geq \frac{n}{2}$ and assume for concreteness that $u_1 \geq u_2 \cdots \geq u_d$. Successively for $k = 1, \ldots, n$ look at the vector

$$\mathbf{u}^{(k)} \overset{\text{def}}{=} \mathbf{u} + \sum_{j=1}^k \mathbf{b}_j = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ u_{k+1} \\ \vdots \\ u_d \end{pmatrix} + \begin{pmatrix} -d+k \\ \vdots \\ -d+k \\ k \\ \vdots \\ k \end{pmatrix} = \begin{pmatrix} u_1-(d-k) \\ \vdots \\ u_k-(d-k) \\ u_{k+1}+k \\ \vdots \\ u_d+k \end{pmatrix}.$$

If for some $1 \leq k \leq n$ we have that $\max \mathbf{u}^{(k)} < \max \mathbf{u}$ then we are done. Otherwise, by our assumption on the order of the coordinates of $\mathbf{u}$ we conclude that $\lceil \mathbf{u}^{(k)} \rceil$ must be obtained at its $k+1$'th coordinate $u_{k+1}+k$ and moreover that $u_1 \leq u_{k+1} + k$. Summing these n inequalities and taking into account that $\mathbf{u} \in \mathbb{R}_0^{n+1}$ we arrive at $(n+1)u_1 \leq \frac{n(n+1)}{2}$, and so $u_1 \leq \frac{n}{2}$. Together with our assumption that $\lceil \mathbf{u} \rceil \geq \frac{n}{2}$ we conclude that in fact $u_1 = \frac{n}{2}$. Thus all the inequalities in the above consideration must be equalities. In particular, $u_1 = u_{k+1} + k$ for all relevant k which implies that $\mathbf{u} = \mathbf{w}$ as desired.

Claim (3) follows from (2) and the following elementary statement about intersections of half-spaces: Given d linear functionals ψ_i on $\mathbb{R}_0^d$ and a point $\mathbf{w} \in \mathbb{R}_0^d$, consider the half spaces

$$H_i^\rho \overset{\text{def}}{=} \left\{ \mathbf{v} \in \mathbb{R}_0^d : \psi_i(\mathbf{v}) \leq \psi_i(\mathbf{w}) + \rho \right\}$$

for $\rho \geq 0$. If the ψ_i are in general position (in fact, if and only if), then the intersection $\cap_1^d H_i^\rho$ is a simplex which is contained in the ball $B_{c\rho}(\mathbf{w})$ for some constant c that depends only on the functionals ψ_i (this constant deteriorates when n of these functionals are almost linearly dependent). In the case at hand the linear functionals are those giving rise for the faces of S_{Φ_*} and $\mathbf{w}$ is a point of $\mathbf{W}_{\Phi_*} + \Delta_*$. In Figure 3 these are the white simplexes centered at the red points. $\square$

Remark 3.9. A nice indirect way to see why there must be equality in Proposition 3.8(2) is as follows. When considering Φ_*, since all the objects involved are invariant under $\mathscr{S}$, it is clear that the established

inclusion implies that the left hand side of (2) is either empty or there is equality. Assuming it is empty, we deduce the existence of a positive $\rho > 0$ such that the left hand side of (3) is empty as well. As will be shown in §4, 5, there are compact A-orbits whose visits to $X^{\geq \delta}$ occur only at times $\mathbf{t}$ belonging to the left hand side of (3). We arrive at the desired contradiction as this set is never empty by Theorem 4.1.

4. Escape and geometry of compact A-orbits

4.1.   Denote
$$\Omega \overset{\text{def}}{=} \{x \in X : Ax \text{ is compact}\}. \tag{4.1}$$
Let U denote the group of diagonal matrices with ± 1 diagonal entries so that UA is the group of all diagonal matrices of determinant ± 1. Let $\varphi : UA \to \mathbb{R}_0^d$ be the homomorphism given by $\varphi(\text{diag}\,(a_1, \ldots, a_d)) = (\log |a_1|, \ldots, \log |a_d|)^t$. Finally, for $x \in \Omega$ we let[5]
$$\Delta_x \overset{\text{def}}{=} \varphi(\text{stab}_{UA}(x)). \tag{4.2}$$
Note that Δ_x might strictly contain $\log(\text{stab}_A(x))$. In practice, $\mathbf{u} \in \Delta_x$ if and only if there is a ± 1 diagonal matrix u such that $ua(\mathbf{u})x = x$ (see (2.1) for the notation), and since U acts by isometries on $\mathbb{R}^d$ and we will only be interested in estimating length of vectors, the presence of u will have no effect.

The results to be proved are stated in an effective way with respect to the covolume $|\Delta_x|$ and thus they become meaningful only when $|\Delta_x| \to \infty$. That this is indeed the case when Ax varies is a consequence of the basic fact from algebraic number theory that says that the regulator of a number field goes to infinity when the number field varies. We take this fact for granted.

A simplex set $\Phi \subset \Delta_x$ will be referred to as a *simplex set for x*. Note that we do not assume that Φ generates Δ_x and in fact, one of our goals (see Theorem 4.14), is to develop tools that will enable to prove that under certain assumptions Φ generates Δ_x; a significant fact from the number theoretical point of view as it allows to compute regulators.

4.2.   In this section, we extract effective information about a compact orbit Ax assuming that (i) x has a very short vector, (ii) there exists a simplex set for x satisfying various assumptions related to the length of the short vector. The results originates from comparing the results of §3 and the following basic fact (recall the notation of (1.1)).

---

[5]Note that $|\Delta_x|$ is (up to a constant factor) the regulator of some order in a totally real number field.

Theorem 4.1. *There exists a universal constant $\delta_0 > 0$ such that for any $x \in X$, $Ax \cap X^{\geq \delta_0} \neq \varnothing$.*

The proof of Theorem 4.1 goes back to the solution of Mordell's inverse problem by Siegel and by Hlawka [Hla50] (see also [Dav36]). The largest possible value of δ_0 (which changes with the choice of norm on $\mathbb{R}^d$) is an interesting quantity in the geometry of numbers. A proof of Margulis to Theorem 4.1 may be found in the appendix of [TW03] and recently it was proved in [McM05], [SW], that for the euclidean norm one can take $\delta_0 \geq 1$.

We begin with interpreting Proposition 3.8(1) in this context.

Lemma 4.2. *Let $x \in \Omega$ and Φ be a simplex set for x. Then $Ax \subset U\left\{a(\mathbf{t})x : \mathbf{t} \in \frac{n}{2}S_\Phi\right\}$.*

Proof. Given any $\mathbf{t} \in \mathbb{R}_0^d$, by Proposition 3.8(1) there exists $\mathbf{u} \in \Delta_x$ with $\mathbf{t}' \stackrel{\text{def}}{=} \mathbf{t} - \mathbf{u} \in \frac{n}{2}S_\Phi$ and so $a(\mathbf{t})x = a(\mathbf{t}')a(\mathbf{u})x = ua(\mathbf{t}')x$ for some $u \in U$. $\square$

Recall the notation of (1.1). The following observation gives a lower bound for $\ell(x)$ in terms of the geometry of a simplex set.

Proposition 4.3. *Let $x \in \Omega$ and let $\Phi \subset \Delta_x$ be a simplex set. Then, $e^{-\frac{n}{2}\xi_\Phi} \ll \ell(x)$.*

Proof. Let $v \in x$ attain the length $\ell(x)$ and let $\mathbf{t} \in \frac{n}{2}S_\Phi$. We give an upper bound for $||a(\mathbf{t})v||$. Write $\mathbf{t} = \frac{n}{2}\sum_1^{n+1} \alpha_j\mathbf{t}_j$ with α_j coefficients of a convex combination. Using the definition of ξ_Φ and the monotonicity of the exponential map we get

$$||a(\mathbf{t})v|| \leq \ell(x) \max_i \left(e^{\sum_j \alpha_j t_{ij}}\right)^{\frac{n}{2}} \leq \ell(x)e^{\frac{n}{2}\xi_\Phi}. \tag{4.3}$$

Theorem 4.1 and Lemma 4.2 imply that there exist $u \in U$ and $\mathbf{t} \in \frac{n}{2}S_\Phi$ such that $\delta_0 \leq ||ua(\mathbf{t})v|| = ||a(\mathbf{t})v||$. This together with (4.3) gives $e^{-\frac{n}{2}\xi_\Phi} \ll \ell(x)$ as desired. $\square$

Motivated by Proposition 4.3 we make the following definition.

Definition 4.4. For $M \geq 1$ say that a simplex set $\Phi \subset \Delta_x$ is *M-tight* for x, if $\ell(x) < Me^{-\frac{n}{2}\xi_\Phi}$. We let

$$\Omega_M \stackrel{\text{def}}{=} \{x \in \Omega : \text{ there exists an } M\text{-tight simplex set for } x\}. \tag{4.4}$$

The following reinterprets Proposition 3.8 in this context and establishes an effective escape of mass statement for lattices in Ω_M in terms of the regulator $|\Delta_x|$. It is the tool with which one can prove Theorem 1.2 directly and not as a corollary to Theorem 1.1.

Proposition 4.5. *Fix $M > 1$ and $\kappa \in (0,1)$ and for $x \in \Omega_M$ with an M-tight simplex set Φ, set*

$$\delta_\kappa(x) = \delta \stackrel{\text{def}}{=} M e^{-\frac{n}{2}|\Delta_x|^{\frac{\kappa}{n}}};$$

$$\rho_\kappa(x) = \rho \stackrel{\text{def}}{=} \frac{|\Delta_x|^{\frac{\kappa}{n}}}{\xi_\Phi} \ll |\Delta_x|^{-\frac{1-\kappa}{n}}; \tag{4.5}$$

$$r_\kappa(x) = r \stackrel{\text{def}}{=} c|\Delta_x|^{\frac{\kappa}{n}} \text{ (c being as in Proposition 3.8).}$$

Then,

(1) $\left\{ a(\mathbf{t})x : \mathbf{t} \in \frac{(1-\rho)n}{2}S_\Phi + \Delta_\Phi \right\} \subset X^{<\delta}$ *(where $X^{<\delta} \stackrel{\text{def}}{=} X \smallsetminus X^{\geq \delta}$).*
(2) $\left\{ \mathbf{t} \in \mathbb{R}_0^{n+1} : a(\mathbf{t}x) \in X^{\geq \delta} \right\} \subset \mathbf{W}'_\Phi + B_r + \Delta_\Phi.$
(3) $\mu_{Ax}(X^{\geq \delta}) \ll |\Delta_x|^{-1+\kappa}.$

Remark 4.6. Hereafter we use the symbols δ, ρ, r as in (4.5) recording implicitly the dependence on x, κ.

Proof. Let $x \in \Omega_M$ and Φ an M-tight simplex set for x. Note that by Lemma 3.5, $\rho \ll |\Delta_x|^{-\frac{1-\kappa}{n}}$.

We prove (1). By the discussion following (4.2) it suffices to prove $\left\{ a(\mathbf{t})x : \mathbf{t} \in \frac{(1-\rho)n}{2}S_\Phi \right\} \subset X^{<\delta}$. Let $\mathbf{t} = (1-\rho)\frac{n}{2}\sum_1^{n+1} \alpha_j \mathbf{t}_j \in (1-\rho)\frac{n}{2}S_\Phi$ (with α_j coefficients of a convex combination). Let $v \in x$ be a vector attaining $\ell(x)$. Similarly to (4.3) we get

$$\|a(\mathbf{t})v\| \leq \ell(x) \max_i \left(e^{\sum_j \alpha_j t_{ij}} \right)^{(1-\rho)\frac{n}{2}}$$

$$< M e^{-\frac{n}{2}\xi_\Phi} \left(e^{\xi_\Phi} \right)^{(1-\rho)\frac{n}{2}} = M e^{-\frac{n}{2}|\Delta_x|^{\frac{\kappa}{n}}} = \delta_\kappa(x), \tag{4.6}$$

which shows that $a(\mathbf{t})x \in X^{<\delta}$.

We prove (2). By (1) we have that

$$\left\{ \mathbf{t} \in \mathbb{R}_0^{n+1} : a(t)x \in X^{\geq \delta} \right\} \subset \mathbb{R}_0^{n+1} \smallsetminus \left((1-\rho)\frac{n}{2}S_\Phi + \Delta_\Phi \right),$$

and by Proposition 3.8(3),

$$\mathbb{R}_0^{n+1} \smallsetminus \left((1-\rho)\frac{n}{2}S_\Phi + \Delta_\Phi \right) \subset \left(\mathbf{W}'_\Phi + B_{c\rho\xi_\Phi} + \Delta_\Phi \right),$$

which implies (2) as $r = c\rho\xi_\Phi$.

Statement (3) now follows from (2): [6]

$$\mu_{Ax}(X^{\geq \delta}) \ll r^n/|\Delta_x| \ll |\Delta_x|^{-1+\kappa}.$$

$\square$

---

[6]Note that the index of $\log(\text{stab}_A(x))$ in Δ_x is bounded by 2^d.

The following result is the one that we use in order to prove Theorem 1.1. It assumes that we have accurate knowledge regarding the position of the orbit Ax at the special times $\mathbf{W}_\Phi$ and concludes that the orbit collapses in an effective manner towards the orbit $A\mathbb{Z}^d$. In order to state it we need the following definition:

Definition 4.7. For $\epsilon > 0, M, C > 1$ we denote by $\Omega_M(\epsilon, C)$ the set of $x \in \Omega_M$ such that there exists an M-tight simplex set Φ for x with respect to which, for any $\tau \in \mathscr{S}$ there exists $g_\tau \in G$, $\mathbf{s}_\tau \in \mathbb{R}_0^d$, such that

$$a(\mathbf{w}_\tau)x = g_\tau a(\mathbf{s}_\tau)\mathbb{Z}^d, \tag{4.7}$$

where g_τ satisfies $||g_\tau - I|| \leq Ce^{-|\Delta_x|^\epsilon}$.

Proposition 4.8. *Fix $M, C > 1$ and $\epsilon > 0$. Then, for any $\kappa < n\epsilon$ there exists $C' > 0$ such that for any $x \in \Omega_M(\epsilon, C)$,*

$$\forall y \in Ax \cap X^{\geq \delta}, \ \mathrm{d}(y, A\mathbb{Z}^d) \leq C'e^{-\frac{1}{2}|\Delta_x|^\epsilon}.$$

Proof. Let $x \in \Omega_M(\epsilon, C)$ and let Φ be an M-tight simplex set as in Definition 4.7. By Proposition 4.5(2) for any point $y \in Ax \cap X^{\geq \delta}$ there exists $\tau \in \mathscr{S}$ such that $y = a(\mathbf{t} + \mathbf{w}_\tau)x$ for some $\mathbf{t} \in \mathbf{w}_\tau + B_r$. By (4.7) it follows that

$$y = a(\mathbf{t})a(\mathbf{w}_\tau)x = a(\mathbf{t})g_\tau a(-\mathbf{t})a(\mathbf{t} + \mathbf{s}_\tau)\mathbb{Z}^d$$

and so the distance between y and $A\mathbb{Z}^d$ is $\leq \mathrm{d}(I, a(\mathbf{t})g_\tau a(-\mathbf{t})) \leq Ce^{2r-|\Delta_x|^\epsilon} = Ce^{2c|\Delta_x|^{\frac{\kappa}{n}}-|\Delta_x|^\epsilon} = Ce^{(-1+2c|\Delta_x|^{\frac{\kappa}{n}-\epsilon})|\Delta_x|^\epsilon}$, and if $\kappa < n\epsilon$ then $2c|\Delta_x|^{\frac{\kappa}{n}-\epsilon} < \frac{1}{2}$ for large values of $|\Delta_x|$ and so for a suitable C' the latter expression is $\leq C'e^{-\frac{1}{2}|\Delta_x|^\epsilon}$ as desired. $\qquad\square$

4.3. In applications one exhibits a simplex set Φ for $x \in \Omega$ and it is desirable to be able to determine that it generates Δ_x. In this section we prove Theorem 4.14 establishing this in some cases. This theorem also discusses the topological complexity of $Ax \cap X^{\geq \delta_1}$.

Our results rely on an assumption regarding the lattice Δ_Φ when considered as a point in the space of lattices in $\mathbb{R}_0^d$ (identified up to homothety). More precisely, we will have to assume that Δ_Φ belongs to a given compact set in that space (it is not clear at the moment if this assumption is necessary). Below we refer to the aforementioned space of lattices as the space of *shapes* in order to distinguish it from X.

It might be worth mentioning here that understanding the set of shapes $\{\Delta_x : x \in \Omega\}$ is a highly non-trivial question with virtually no results (although in the next section we will see that the shape Δ_* is an accumulation point of this set).

Definition 4.9. Given $M > 1$ and K a compact set of shapes we denote by $\Omega_{M,K}$ the set of $x \in \Omega_M$ possessing an M-tight simplex set Φ for x with shape $\Delta_\Phi \in K$.

Because Δ_Φ is obtained from Δ_* by applying a distortion map h_Φ followed by a dilation, it is obvious from the definition that the set $\{\Delta_\Phi : ||h_\Phi|| \le T\}$ is contained in some compact set $K = K_T$ of shapes. Lemma 4.10 below is a converse statement of some sort for tight simplex sets Φ with Δ_Φ in a given compact set of shapes.

Lemma 4.10. *For any $M > 0$ and any compact set of shapes K, if $x \in \Omega_{M,K}$ and $\Phi \subset \Delta_x$ is an M-tight simplex set then the distortion map of Φ satisfies $||h_\Phi|| \ll_{M,K} 1$.*

We note that in applications, most often one knows that $x \in \Omega_M$ belongs to $\Omega_{M,K}$ because one exhibits a simplex set Φ of bounded distortion, which makes the conclusion of the lemma automatic. Nonetheless we chose to give it here as it seems reasonable to expect an application in which one knows that Δ_Φ belongs to a compact set of shapes without knowing that Φ is responsible for that.

Proof. Let Φ be as in the statement. The assumption $\Delta_\Phi \in K$, implies that there exists a basis $\{\mathbf{t}_\ell\}_1^n$ of Δ_Φ such that the linear map h sending $\mathbf{b}_\ell$ to $|\Delta_\Phi|^{-\frac{1}{n}}\mathbf{t}_\ell$, ($\ell = 1 \dots n$), satisfies $||h|| \asymp_K 1$. Denote $\Phi' \stackrel{\text{def}}{=} \{\mathbf{t}_\ell\}_1^{n+1}$ where $\mathbf{t}_{n+1} = -\sum_1^n \mathbf{t}_\ell$ so that $h_{\Phi'} = h$. Using Lemma 3.4(1) twice we get

$$1 \asymp_K ||h_{\Phi'}|| \asymp |\Delta_\Phi|^{-\frac{1}{n}}\xi_{\Phi'} \implies |\Delta_\Phi|^{\frac{1}{n}} \asymp_K \xi_{\Phi'} \implies$$

$$||h_\Phi|| \asymp |\Delta_\Phi|^{-\frac{1}{n}}\xi_\Phi \asymp_K \frac{\xi_\Phi}{\xi_{\Phi'}}.$$

On the other hand, by Proposition 4.3 and our assumption we have that $e^{-\frac{n}{2}\xi_{\Phi'}} \ll \ell(x) \le Me^{-\frac{n}{2}\xi_\Phi}$ and therefore $\xi_\Phi - \xi_{\Phi'} \ll_M 1$. We conclude that $||h_\Phi|| \ll_{M,K} 1 + \frac{1}{\xi_{\Phi'}}$. Since $\xi_{\Phi'} \gg 1$ by Lemma 3.5 and the remark made at the beginning of this section saying that $|\Delta_x| \gg 1$, we get $||h_\Phi|| \ll_{M,K} 1$ as desired. $\square$

The following lemma gives us the tool with which we bound the index of one lattice in another.

Lemma 4.11. *Let $\Delta < \Sigma$ be lattices and suppose that there is a ball B and vectors $\mathbf{v}_i$, $i = 1, \dots, \ell$, such that (i) $\Sigma \subset \{\mathbf{v}_i\}_{i=1}^\ell + B + \Delta$, and (ii) $|\Sigma \cap (\mathbf{v}_i + B + \mathbf{u})| \le 1$ for any $\mathbf{u} \in \Delta$, $1 \le i \le \ell$. Then $[\Sigma : \Delta] \le \ell$.*

Proof. Using assumption (i) we may define a map $\alpha : \Sigma \to \Delta$ by the following procedure: Given $\mathbf{t} \in \Sigma$ we choose $1 \le i \le \ell$ such that

$\mathbf{t} \in \mathbf{v}_i + B + \mathbf{u}$ (with $\mathbf{u} \in \Delta$), and set $\alpha(\mathbf{t}) = \mathbf{u}$. Assumption (ii) implies that α is at most ℓ to 1. Note also that there is a constant T_0 such that $||\alpha(\mathbf{t})|| \leq ||\mathbf{t}|| + T_0$. We conclude that if we denote by C_T the cube of side length T centered at 0, then for any $T > 0$, $|\Sigma \cap C_T| \leq \ell|\alpha(\Sigma) \cap C_{T+T_0}|$.

Recall that for any lattice Δ', $|\Delta'|^{-1} = \lim_T \frac{1}{T^n}|\Delta' \cap C_T|$. It follows that

$$[\Sigma : \Delta] = \frac{|\Delta|}{|\Sigma|} = \lim_T \frac{|\Sigma \cap C_T|}{|\Delta \cap C_T|} \leq \limsup_T \ell \frac{|\alpha(\Sigma) \cap C_{T+T_0}|}{|\Delta \cap C_T|} \leq \ell.$$

$\square$

We use the following definition to discuss what we referred to above as 'topological complexity'.

Definition 4.12. Given $\delta_1 > 0$ and a lattice x we say that $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}_0^d$ are two *distinct visits times* to $X^{\geq \delta_1}$ if $a(\mathbf{t}_i)x$ belong to two distinct connected components of $Ax \cap X^{\geq \delta_1}$.

The following lemma will help us distinguish between visit times.

Lemma 4.13. *Let $x \in \Omega$, B_1, B_2 two balls in $\mathbb{R}_0^d$ and $\delta_1 > 0$. Assume that (i) $\exists \mathbf{t}_i \in B_i$ with $a(\mathbf{t}_i)x \in X^{\geq \delta_1}$, and (ii) $\{a(\mathbf{t})x : \mathbf{t} \in \partial B_1 \cup \partial B_2\} \subset X^{< \delta_1}$. Then, if $\mathbf{t}_i$ are not distinct visit times to $X^{\geq \delta_1}$ then there exist $\mathbf{u}_i \in B_i$ such that $\mathbf{u}_1 - \mathbf{u}_2 \in \Delta_x$.*

Proof. If $a(\mathbf{t}_i)x$ are in the same component of $Ax \cap X^{\geq \delta_1}$ then we may join them by a path within that component. As the map $\mathbf{t} \mapsto a(\mathbf{t})x$ is a covering map, this path can be lifted to a one starting at $\mathbf{t}_1$. Because of assumption (ii), this lift is contained in B_1 and in particular, its end point $\mathbf{t}_2'$, which satisfies $a(\mathbf{t}_2')x = a(\mathbf{t}_2)x$, is in B_1. We conclude that the difference $\mathbf{t}_2' - \mathbf{t}_2$ belongs to Δ_x and so we may take $\mathbf{u}_1 = \mathbf{t}_2'$ and $\mathbf{u}_2 = \mathbf{t}_2$. $\square$

Theorem 4.14. *Fix $M > 1$, K a compact set of shapes, and $\delta_1 \in (0, \delta_0)$. For all but finitely many $x \in \Omega_{M,K}$, if $\Phi \subset \Delta_x$ is an M-tight simplex set and we let*

$$\mathbf{W}_\Phi'' \stackrel{\mathrm{def}}{=} \left\{ \mathbf{w}_\tau \in \mathbf{W}_\Phi' : \exists \mathbf{t} \in \mathbf{w}_\tau + B_r, \; a(\mathbf{t})x \in X^{\geq \delta_1} \right\},$$

then,

    (1) $[\Delta_x : \Delta_\Phi] \leq |\mathbf{W}_\Phi''|$ *(and of course, $|\mathbf{W}_\Phi''| \leq n!$).*
    (2) *If d is prime then $[\Delta_x : \Delta_\Phi] = 1$.*
    (3) *If $\mathbf{W}_\Phi'' = \mathbf{W}_\Phi'$ then $[\Delta_x : \Delta_\Phi] = 1$.*
    (4) *Under assumption (2) or (3), there are at least $|\mathbf{W}_\Phi''|$ distinct visits to $X^{\geq \delta_1}$.*

Remark 4.15. We note that the assumption in (3) yields the best results, namely, we both get that Φ generates Δ_x and that the orbit Ax is as complex as it could be. We point out two ways to verify this assumption in practice:

(1) If the lattice x belongs to $\Omega_M(\epsilon, C)$ and the times $\mathbf{s}_\tau$ from Definition 4.7 are assumed to be bounded then it follows that for any $\mathbf{w}_\tau \in \mathbf{W}_\Phi$, $a(\mathbf{w}_\tau)x \in X^{\geq \delta_1}$ for some fixed δ_1. This is what happens in the construction discussed in §5 below.

(2) Sometimes one can exploit symmetries of x to bootstrap the existence of one $\mathbf{w}_\tau$ such that $a(\mathbf{w}_\tau)x \in X^{\geq \delta_1}$ and upgrade it into the same statement for all $\mathbf{w}_\tau$. Such symmetries might exist if one assumes x arises from a Galois extension.

Proof. Let $x \in \Omega_{M,K}$ and Φ an M-tight simplex set for x. We use the notation of Proposition 4.5 fixing some $\kappa \in (0, 1)$.

Consider a time $\mathbf{t}$ such that $a(\mathbf{t})x \in X^{\geq \delta_0}$. By Proposition 4.5(2), (and the definition of $\mathbf{W}_\Phi''$), for all but finitely many $x \in \Omega_M$ we have that

$$\mathbf{t} + \Delta_x \subset \mathbf{W}_\Phi'' + B_r + \Delta_\Phi, \tag{4.8}$$

where $r \asymp |\Delta_x|^{\frac{\kappa}{n}}$ is given in (4.5). Statement (1) will follow from Lemma 4.11, once we prove that for any τ, the coset $\mathbf{t} + \Delta_x$ can contain at most one point in each ball $\mathbf{w}_\tau + B_r + \mathbf{v}$ (for $\mathbf{v} \in \Delta_\Phi$).

Assume such a ball contains two points of a coset of Δ_x. Then Δ_x contains a non-trivial vector of size $\leq 2r$. The assumption that the shape of Δ_Φ is in K implies, by Lemma 4.10, that the distortion map h_Φ satisfies $||h_\Phi|| \ll_{M,K} 1$, which in turn, by Lemma 3.4(2), implies that the distances between the balls composing $\mathbf{W}_\Phi + B_r + \Delta_\Phi$ is $\gg_{M,K} |\Delta_\Phi|^{\frac{1}{n}} - 2r$. Since the latter expression is $> 2r$ as soon as $|\Delta_x|$ is big enough, we arrive at a contradiction to the containment (4.8).

We now establish (2) and (3). Let $\mathbf{t}$ be such that $a(\mathbf{t})x \in X^{\geq \delta_0}$ as above. By (4.8), we may assume that

$$\mathbf{t} \in \mathbf{w}_\tau + B_r, \tag{4.9}$$

for some $\mathbf{w}_\tau \in \mathbf{W}_\Phi''$. We will show that $\mathbf{t} + \Delta_x \subset \mathbf{w}_\tau + B_r + \Delta_\Phi$ which together with the already established fact by which each ball in the latter set contains at most one point of $\mathbf{t} + \Delta_x$, will imply, by Lemma 4.11, that $\Delta_x = \Delta_\Phi$ as claimed.

Assuming otherwise, there exists $\mathbf{w}_\tau \neq \mathbf{w}_\sigma \in \mathbf{W}_\Phi''$ such that $(\mathbf{t} + \Delta_x) \cap (\mathbf{w}_\sigma + B_r) \neq \varnothing$, which together with (4.9) implies that

$$\exists \mathbf{s} \in B_{2r} \text{ such that } \mathbf{v} \stackrel{\text{def}}{=} \mathbf{w}_\tau - \mathbf{w}_\sigma + \mathbf{s} \in \Delta_x. \tag{4.10}$$

By Lemma 3.7(2) we conclude the existence of $\mathbf{w}_{\tau'} \in \mathbf{W}'_\Phi$ and an integer $1 \leq k \leq n$ such that

$$\mathbf{w}_{\tau'} + k(\mathbf{w}_\tau - \mathbf{w}_\sigma) \notin \mathbf{W}_\Phi + \Delta_\Phi. \tag{4.11}$$

Under assumption (3), $\mathbf{w}_{\tau'} \in \mathbf{W}''_\Phi$ automatically. Under the primality assumption (2), Lemma 3.7(2) says that we may choose $\tau' = \tau$ and hence in both cases we may assume $\mathbf{w}_{\tau'} \in \mathbf{W}''_\Phi$. By definition this means that

$$\exists \mathbf{t}' \in \mathbf{w}_{\tau'} + B_r \text{ such that } a(\mathbf{t}')x \in X^{\geq \delta_1}. \tag{4.12}$$

As the objects in (4.11) are obtained from the corresponding ones for Φ_* by applying a distortion map h_Φ followed by a dilation by a factor of $|\Delta_\Phi|^{\frac{1}{n}}$, we conclude by Lemma 4.10, Lemma 3.4(2), that

$$\mathrm{d}(\mathbf{w}_{\tau'} + k(\mathbf{w}_\tau - \mathbf{w}_\sigma), \mathbf{W}_\Phi + \Delta_\Phi) \gg_{M,K} |\Delta_x|^{\frac{1}{n}}.$$

By (4.10), (4.12), $\mathbf{w}_{\tau'} + k(\mathbf{w}_\tau - \mathbf{w}_\sigma) = \mathbf{t}' + k\mathbf{v} + \mathbf{s}'$ for $\mathbf{s}'$ with $||\mathbf{s}'|| \ll r$, and therefore we have that $\mathrm{d}(\mathbf{t}' + k\mathbf{v}, \mathbf{W}_\Phi + \Delta_\Phi) \gg_{M,K} |\Delta_x|^{\frac{1}{n}}$ which shows in particular, that for all but finitely many $x \in \Omega_{M,K}$, $\mathbf{t}' + k\mathbf{v} \notin \mathbf{W}_\Phi + B_r + \Delta_\Phi$. In turn, Proposition 4.5(2) gives that $a(\mathbf{t}' + k\mathbf{v})x \in X^{<\delta}$, which contradicts (4.10), (4.12), from which it follows that $a(\mathbf{t}' + k\mathbf{v})x = ua(\mathbf{t})x \in X^{\geq \delta_1}$ (for some $u \in U$).

Statement (4) follows from Lemma 4.13 and the above considerations: For any $\mathbf{w}_\tau \in \mathbf{W}''_\Phi$ let $\mathbf{s}_\tau \in B_r$ be such that $a(\mathbf{w}_\tau + \mathbf{s}_\tau)x \in X^{\geq \delta_1}$. By Lemma 4.13, if amongst these times are two non-distinct visit times, then Δ_x must contain a vector as in (4.10) which results in a contradiction as shown above. $\qquad\square$

4.4.   We summarize Proposition 4.5, Proposition 4.8, and Theorem 4.14 for convenience of reference:

Corollary 4.16. *Fix $M, C > 1$, $\epsilon > 0$ and K a compact set of shapes. Then the following hold:*

(1) *By Proposition 4.5, any sequence of (distinct) orbits Ax_k of lattices $x_k \in \Omega_M$ satisfies the conclusion of Corollary 1.2.*

(2) *By Proposition 4.8, any sequence of (distinct) orbits Ax_k of lattices $x_k \in \Omega_M(\epsilon, C)$ satisfies the conclusion of Theorem 1.1.*

(3) *By Theorem 4.14, for any sequence of (distinct) orbits Ax_k of lattices $x_k \in \Omega_M(\epsilon, C) \cap \Omega_{M,K}$, if Φ_k is an M-tight simplex set for x_k satisfying Definition 4.7 such that the times $\mathbf{s}_\tau$ appearing there are uniformly bounded, then, for any $\delta_1 \in (0, \delta_0)$, and for all large k, (i) Φ_k generates Δ_x and, (ii) $Ax_k \cap X^{\geq \delta_1}$ has at least $n!$ distinct connected components.*

5. THE CONSTRUCTION

In this section we construct infinite families of lattices satisfying the requirements of Corollary 4.16 and doing so, we prove Theorem 1.1.

5.1. **The polynomials.** Fix $\eta > 0$ and define

$$\mathbb{Z}^d(\eta) \stackrel{\text{def}}{=} \left\{ \mathbf{m} \in \mathbb{Z}^d : \forall i, j, \ \frac{|m_i - m_j|}{||\mathbf{m}||} \geq \eta \right\}. \tag{5.1}$$

Thus, $\mathbb{Z}^d(\eta)$ consists of those integer vectors which are projectively bounded away from the $\binom{d}{2}$ hyperplanes of equal coordinates. For $\mathbf{m} \in \mathbb{Z}^d(\eta)$ define

$$p_{\mathbf{m}}(x) \stackrel{\text{def}}{=} \prod_1^d (x - m_j) - 1.$$

Lemma 5.1. *Fix $\eta > 0$. For all but finitely many $\mathbf{m} \in \mathbb{Z}^d(\eta)$, the polynomial $p_{\mathbf{m}}$ is irreducible over $\mathbb{Q}$ and has d real roots $\theta_j = \theta_j(\mathbf{m})$, $j = 1 \ldots d$ which when put in the correct order satisfy $\theta_j = m_j + O_\eta(||\mathbf{m}||^{-n})$.*

Proof. Let $\mathbf{m} \in \mathbb{Z}^d(\eta)$ and let $\theta \in \mathbb{C}$ be a root of $p_{\mathbf{m}}$; that is,

$$\prod_{j=1}^d (\theta - m_j) = 1. \tag{5.2}$$

As $|m_i - m_j| \geq \eta||\mathbf{m}||$, for all but finitely many $\mathbf{m}$, θ can satisfy $|\theta - m_j| < 1$ for at most one choice of j. In turn, because of the above equation, this inequality must hold for exactly one such $j \stackrel{\text{def}}{=} j_\theta$. It follows that for $j \neq j_\theta$ we have $|\theta - m_j| = |m_{j_\theta} - m_j| + O(1) \asymp_\eta ||\mathbf{m}||$. Going back to the equation $|\theta - m_{j_\theta}| = \prod_{j \neq j_\theta} |\theta - m_j|^{-1}$ we conclude that $|\theta - m_{j_\theta}| \asymp_\eta ||\mathbf{m}||^{-n}$.

We claim that the map $\theta \mapsto j_\theta$ is one to one and onto from the set of complex zeros of $p_{\mathbf{m}}$ to $\{1, \ldots, d\}$. Assume this is not the case and write $p_{\mathbf{m}}(x) = \prod_{j=1}^d (x - \theta_j)$. Then there are $j_1 \neq j_2$ such that $\ell \stackrel{\text{def}}{=} j_{\theta_{j_1}} = j_{\theta_{j_2}}$. We conclude that the value

$$|p_{\mathbf{m}}(m_\ell)| = \prod_{j=1}^d |m_\ell - \theta_j| \tag{5.3}$$

is a non-zero integer on the one hand, but on the other hand, the two factors in the product (5.3) that correspond to j_1, j_2 are $O_\eta(||\mathbf{m}||^{-n})$ while all the other $n - 1$ factors are $O_\eta(||\mathbf{m}||)$. We conclude that for all but finitely many $\mathbf{m}$, the value in (5.3) is < 1 which gives a contradiction.

Thus, we may order the roots θ_j of $p_{\mathbf{m}}$ in such a way so that

$$|\theta_j - m_j| \asymp_\eta ||\mathbf{m}||^{-n}, \tag{5.4}$$

$$\text{for } \ell \neq j, \ |\theta_j - m_\ell| \asymp_\eta ||\mathbf{m}||. \tag{5.5}$$

This immediately implies that (as soon as $||\mathbf{m}||$ is large enough), all the roots are real because $p_{\mathbf{m}}$ is real so complex roots come in conjugate pairs.

A similar argument gives the irreducibility of $p_{\mathbf{m}}$ over $\mathbb{Q}$: If $p_{\mathbf{m}}$ is reducible over $\mathbb{Q}$ then by Gauss' lemma it must be reducible over $\mathbb{Z}$. Hence, there must exist a proper subset I of $\{1 \dots d\}$ such that $q(x) = \prod_{j \in I}(x - \theta_j)$ is a polynomial over the integers. We choose $\ell \in I$ and consider the value $|q(m_\ell)| = \prod_{j \in I}|m_\ell - \theta_j|$. This number should be a (non-zero) integer by assumption but on the other hand, the term in the product which corresponds to $j = \ell$ is $O_\eta(||\mathbf{m}||^{-n})$ while all the other terms are $O_\eta(||\mathbf{m}||)$. We derive a contradiction (as soon as $||\mathbf{m}||$ is large enough), because there are at most $n - 1$ such terms (as the cardinality of I is assumed to be $\leq n$). $\qquad\square$

5.2. **The lattices.** Fix $\eta > 0$ and $\mathbf{m} \in \mathbb{Z}^d(\eta)$ with $||\mathbf{m}||$ large enough so that Lemma 5.1 applies. Fix an abstract root $\theta = \theta(\mathbf{m})$ of $p_{\mathbf{m}}$ and let $F_{\mathbf{m}} \stackrel{\text{def}}{=} \mathbb{Q}(\theta)$ be the number field generated by it. By Lemma 5.1, $F_{\mathbf{m}}$ is a totally real number field of degree d over $\mathbb{Q}$. By Lemma 5.1, we can order the embeddings $\sigma_1 \dots \sigma_d$ of $F_{\mathbf{m}}$ into $\mathbb{R}$ in such a way so that $\theta_j \stackrel{\text{def}}{=} \sigma_j(\theta)$ satisfies $\theta_j = m_j + O_\eta(||\mathbf{m}||^{-n})$. Let

$$\Lambda_{\mathbf{m}} = \text{span}_{\mathbb{Z}}\{1, \theta, \dots, \theta^n\} \subset F_{\mathbf{m}}.$$

Then, $\Lambda_{\mathbf{m}}$ is a full module in $F_{\mathbf{m}}$; that is, it is a free $\mathbb{Z}$-module of rank d that spans $F_{\mathbf{m}}$ over $\mathbb{Q}$ (in fact it is the order $\mathbb{Z}[\theta]$). Let $\boldsymbol{\sigma} : F_{\mathbf{m}} \hookrightarrow \mathbb{R}^d$ denote the map whose j'th coordinate is σ_j. Then it is a well known fact that the image under $\boldsymbol{\sigma}$ of a full module in $F_{\mathbf{m}}$ is a lattice in $\mathbb{R}^d$. In particular, if $D_{\mathbf{m}} \stackrel{\text{def}}{=} \text{covol}(\boldsymbol{\sigma}(\Lambda_{\mathbf{m}}))$ then

$$x_{\mathbf{m}} \stackrel{\text{def}}{=} D_{\mathbf{m}}^{-\frac{1}{d}}\boldsymbol{\sigma}(\Lambda_{\mathbf{m}})$$

is a unimodular lattice in $\mathbb{R}^d$; i.e. $x_{\mathbf{m}} \in X$.

5.3. We now show that Corollary 4.16 applies for the lattices constructed above and by that complete the proof of the statements in §1.

Proposition 5.2. *For any $\eta > 0$ there exists $M, C > 1$, $\epsilon > 0$ and a compact set of shapes K such that for all but finitely many $\mathbf{m} \in \mathbb{Z}^d(\eta)$, the lattice $x_{\mathbf{m}}$ belongs to $\Omega_M(\epsilon, C) \cap \Omega_{M,K}$. Moreover, we exhibit an*

M-tight simplex set $\Phi_{\mathbf{m}}$ for $x_{\mathbf{m}}$ satisfying Definition 4.7 such that the times $\mathbf{s}_\tau$ appearing there are uniformly bounded.

Proof. Let $\mathbf{m} \in \mathbb{Z}^d(\eta)$ be of large enough norm so that Lemma 5.1 applies. Our goal is to find a simplex set Φ for $x_{\mathbf{m}}$ that will be M-tight and will satisfy the requirements of $\Omega_M(\epsilon, C)$ and $\Omega_{M,K}$. We start by estimating $\ell(x_{\mathbf{m}})$.

As $\mathbf{1} \in \boldsymbol{\sigma}(\Lambda_{\mathbf{m}})$ we have that $v_{\mathbf{m}} \overset{\text{def}}{=} D_{\mathbf{m}}^{-\frac{1}{d}} \mathbf{1} \in x_{\mathbf{m}}$. Let us calculate $D_{\mathbf{m}}$. By definition $\boldsymbol{\sigma}(\Lambda_{\mathbf{m}})$ is the lattice spanned by the columns of the (Vandermonde) matrix $(\sigma_i(\theta^{j-1})) = (\theta_i^{j-1})$;

$$D_{\mathbf{m}} = \det \begin{pmatrix} 1 & \theta_1 & \theta_1^2 & \dots & \theta_1^n \\ 1 & \theta_2 & \theta_2^2 & \dots & \theta_2^n \\ & & \ddots & & \\ 1 & \theta_d & \theta_d^2 & \dots & \theta_d^n \end{pmatrix} = \prod_{i<j}(\theta_j - \theta_i) = \prod_{i<j}(m_j - m_i + O_\eta(||\mathbf{m}||^{-n}))$$

$$= ||\mathbf{m}||^{\binom{d}{2}} \prod_{i<j}(\frac{m_j - m_i}{||\mathbf{m}||} + O_\eta(||\mathbf{m}||^{-d}))$$

$$= ||\mathbf{m}||^{\binom{d}{2}} O(1). \tag{5.6}$$

We conclude that

$$\ell(x_{\mathbf{m}}) \leq ||v_{\mathbf{m}}|| \ll ||\mathbf{m}||^{-\frac{n}{2}}. \tag{5.7}$$

We now turn to investigate $\Delta_{x_{\mathbf{m}}}$ in order to find a suitable simplex set Φ in it. The link with the action of the diagonal group on X originates from the following observation. Let $\alpha \in F_{\mathbf{m}}$ and let $R_\alpha : F_{\mathbf{m}} \to F_{\mathbf{m}}$ denote multiplication by α. Then the following diagram commutes:

$$\begin{array}{ccc} F_{\mathbf{m}} & \overset{\boldsymbol{\sigma}}{\longrightarrow} & \mathbb{R}^d \\ {\scriptstyle R_\alpha}\downarrow & & \downarrow{\scriptstyle \text{diag}(\boldsymbol{\sigma}(\alpha))} \\ F_{\mathbf{m}} & \underset{\boldsymbol{\sigma}}{\longrightarrow} & \mathbb{R}^d \end{array} \tag{5.8}$$

It is clear that for any $1 \leq \ell \leq d$, $\omega_\ell \Lambda_{\mathbf{m}} \subset \Lambda_{\mathbf{m}}$, where

$$\omega_\ell = \omega_\ell(\mathbf{m}) \overset{\text{def}}{=} \theta - m_\ell. \tag{5.9}$$

In fact, we have that $\omega_\ell \Lambda_{\mathbf{m}} = \Lambda_{\mathbf{m}}$ because ω_ℓ is a unit in the ring of integers. To see this, note that (5.2) shows that both ω_ℓ and its inverse are algebraic integers. It follows from (5.8) that $\text{diag}(\boldsymbol{\sigma}(\omega_\ell)) \in UA$ stabilizes $x_{\mathbf{m}}$ and therefore, by (4.2),

$$\mathbf{t}_\ell \overset{\text{def}}{=} \log(|\text{diag}(\boldsymbol{\sigma}(\omega_\ell))|) \in \Delta_{x_{\mathbf{m}}}. \tag{5.10}$$

We claim that $\Phi_{\mathbf{m}} = \Phi \overset{\text{def}}{=} \{\mathbf{t}_\ell\}_{\ell=1}^d$ forms the desired simplex set. The fact that $\sum_\ell \mathbf{t}_\ell = 0$ follows from (5.2). From (5.4), (5.5) we conclude several things: First we note that the set $\frac{1}{\log(||\mathbf{m}||)}\Phi_{\mathbf{m}}$ converges as $||\mathbf{m}|| \to \infty$ to Φ_*. This implies in particular, that Φ spans $\mathbb{R}_0^d$ and in turn implies that it is a simplex set. Moreover, it shows that the distortion map h_Φ converges to the identity and so there exists a compact set K of shapes such that $\Delta_\Phi \in K$. Second, we conclude that

$$\xi_\Phi = \max_{1 \le \ell \le d} \lceil \mathbf{t}_\ell \rceil \le \log ||\mathbf{m}|| + O_\eta(1). \tag{5.11}$$

Combining (5.7) and (5.11) we conclude one can choose $M = M(\eta)$ such that $\ell(x_{\mathbf{m}}) < Me^{-\frac{n}{2}\xi_\Phi}$. Summarizing, we have established that for all but finitely many $\mathbf{m} \in \mathbb{Z}^d(\eta)$, $x_{\mathbf{m}} \in \Omega_{M,K}$ and that $\Phi_{\mathbf{m}}$ is an M-tight simplex set for $x_{\mathbf{m}}$ with Δ_Φ in K.

We now turn to study the lattices $a(\mathbf{w}_\tau)x_{\mathbf{m}}$ for $\tau \in \mathscr{S}$ in order to verify Definition 4.7.

It would be convenient to introduce the following notation: Given two matrices $A = (a_{ij}), B = (b_{ij})$ (say, depending on the parameter η), we write

$$A \ll_\eta^{\text{av}} B$$

to indicate that there exists a constant $c = c(\eta) > 0$, such that $|a_{ij}| \le c|b_{ij}|$.

For simplicity let us assume that τ is the identity permutation and denote $\mathbf{w} \overset{\text{def}}{=} \mathbf{w}_{\text{id}}^{\Phi_{\mathbf{m}}}$. We begin by estimating $a(\mathbf{w})$ and then continue to find a suitable matrix representing the lattice $a(\mathbf{w})x_{\mathbf{m}}$: By definition (3.2),

$$\mathbf{w} = \sum_{\ell=1}^d \frac{\ell-1}{d} \cdot \mathbf{t}_\ell$$

and so by (5.10),

$$a(\mathbf{w}) = \prod_{\ell=1}^d \left(\text{diag}\left(|\sigma_1(\omega_\ell)|, \ldots, |\sigma_d(\omega_\ell)|\right)\right)^{\frac{\ell-1}{d}}$$

$$= \text{diag}\left(|\sigma_1(\prod_{\ell=1}^d \omega_\ell^{\ell-1})|, \ldots, |\sigma_d(\prod_{\ell=1}^d \omega_\ell^{\ell-1})|\right)^{\frac{1}{d}}$$

$$\ll_\eta^{\text{av}} ||\mathbf{m}||^{-\frac{n}{2}} \text{diag}\left(||\mathbf{m}||^n, ||\mathbf{m}||^{n-1}, \ldots, ||\mathbf{m}||, 1\right), \tag{5.12}$$

where the last estimate follows from (5.9), (5.4), (5.5).

Recall that the lattice $x_{\mathbf{m}}$ has a basis corresponding to the basis of $\Lambda_{\mathbf{m}}$ given by $\{\theta^j\}$, $j = 0 \ldots n$. We make the following change of base of $\Lambda_{\mathbf{m}}$

$$\{1, \theta, \ldots, \theta^n\} \to \{1, \omega_1, \omega_1\omega_2, \ldots, \omega_1 \cdots \omega_n\}.$$

The corresponding basis of $x_{\mathbf{m}}$ is given by the columns of the matrix

$$f_{\mathbf{m}} \overset{\text{def}}{=} D_{\mathbf{m}}^{-\frac{1}{d}} \begin{pmatrix} 1 & \sigma_1(\omega_1) & \dots & \sigma_1(\omega_1 \cdots \omega_n) \\ 1 & \sigma_2(\omega_1) & \dots & \sigma_2(\omega_1 \cdots \omega_n) \\ \vdots & & & \vdots \\ 1 & \sigma_d(\omega_1) & \dots & \sigma_d(\omega_1 \cdots \omega_n) \end{pmatrix}$$

Recall that by (5.6), $D_{\mathbf{m}}^{-\frac{1}{d}} = O(||\mathbf{m}||^{-\frac{n}{2}})$ and by (5.4), (5.5) that

$$\sigma_i(\omega_\ell) = \begin{cases} O_\eta(||\mathbf{m}||) & \text{if } \ell \neq i; \\ O_\eta(||\mathbf{m}||^{-n}) & \text{if } \ell = i \end{cases} \implies \sigma_i(\prod_1^j \omega_\ell) = \begin{cases} O_\eta(||\mathbf{m}||^{j-1-n}) & \text{if } i \leq j; \\ O_\eta(||\mathbf{m}||^j) & \text{if } i > j. \end{cases}$$

In other words,

$$f_{\mathbf{m}} \ll_\eta^{\text{av}} ||\mathbf{m}||^{-\frac{n}{2}} \begin{pmatrix} 1 & ||\mathbf{m}||^{-n} & ||\mathbf{m}||^{-(n-1)} & ||\mathbf{m}||^{-(n-2)} & \dots & ||\mathbf{m}||^{-1} \\ 1 & ||\mathbf{m}|| & ||\mathbf{m}||^{-(n-1)} & ||\mathbf{m}||^{-(n-2)} & \dots & ||\mathbf{m}||^{-1} \\ 1 & ||\mathbf{m}|| & ||\mathbf{m}||^2 & ||\mathbf{m}||^{-(n-2)} & \dots & ||\mathbf{m}||^{-1} \\ 1 & ||\mathbf{m}|| & ||\mathbf{m}||^2 & ||\mathbf{m}||^3 & \dots & ||\mathbf{m}||^{-1} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 1 & ||\mathbf{m}|| & ||\mathbf{m}||^2 & ||\mathbf{m}||^3 & \dots & ||\mathbf{m}||^n \end{pmatrix}. \quad (5.13)$$

So, by (5.12), (5.13) we see that the matrix $a(\mathbf{w})f_{\mathbf{m}}$ which represents $a(\mathbf{w})x_{\mathbf{m}}$ satisfies

$$a(\mathbf{w})f_{\mathbf{m}} \ll_\eta^{\text{av}} \begin{pmatrix} 1 & ||\mathbf{m}||^{-n} & ||\mathbf{m}||^{-(n-1)} & ||\mathbf{m}||^{-(n-2)} & \dots & ||\mathbf{m}||^{-1} \\ ||\mathbf{m}||^{-1} & 1 & ||\mathbf{m}||^{-n} & ||\mathbf{m}||^{-(n-1)} & \dots & ||\mathbf{m}||^{-2} \\ ||\mathbf{m}||^{-2} & ||\mathbf{m}||^{-1} & 1 & ||\mathbf{m}||^{-n} & \dots & ||\mathbf{m}||^{-3} \\ ||\mathbf{m}||^{-3} & ||\mathbf{m}||^{-2} & ||\mathbf{m}||^{-1} & 1 & \dots & ||\mathbf{m}||^{-4} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ ||\mathbf{m}||^{-n} & ||\mathbf{m}||^{-(n-1)} & ||\mathbf{m}||^{-(n-2)} & ||\mathbf{m}||^{-(n-3)} & \dots & 1 \end{pmatrix}. \quad (5.14)$$

Write $a(\mathbf{w})f_{\mathbf{m}} = (c_{ij})$. Working with the inequalities in (5.14) and the fact that $\det a(\mathbf{w})f_{\mathbf{m}} = 1$ we deduce that:

(1) For $\ell = 1 \dots d$, $|c_{\ell\ell}| \ll_\eta 1$.
(2) For $i \neq j$, $|c_{ij}| \ll_\eta ||\mathbf{m}||^{-1}$.
(3) $1 = \det(c_{ij}) = \prod_{\ell=1}^d c_{\ell\ell} + O_\eta(||\mathbf{m}||^{-1})$.

Consider now the brute force decomposition

$$(c_{ij}) = \begin{pmatrix} \frac{c_{11}}{c_{11}}(\prod_1^d c_{\ell\ell})^{\frac{1}{d}} & \dots & \frac{c_{1d}}{c_{dd}}(\prod_1^d c_{\ell\ell})^{\frac{1}{d}} \\ \vdots & \frac{c_{ij}}{c_{jj}}(\prod_1^d c_{\ell\ell})^{\frac{1}{d}} & \vdots \\ \frac{c_{d1}}{c_{11}}(\prod_1^d c_{\ell\ell})^{\frac{1}{d}} & \dots & \frac{c_{dd}}{c_{dd}}(\prod_1^d c_{\ell\ell})^{\frac{1}{d}} \end{pmatrix} \begin{pmatrix} \frac{c_{11}}{(\prod_1^d c_{\ell\ell})^{\frac{1}{d}}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{c_{dd}}{(\prod_1^d c_{\ell\ell})^{\frac{1}{d}}} \end{pmatrix}.$$

If we denote the matrices on the right hand side by $g, a(\mathbf{s})u$ respectively (where u is the appropriate ± 1 diagonal matrix), then it follows from (1), (2), (3) that $||g - I|| \ll_\eta ||\mathbf{m}||^{-1}$, $||\mathbf{s}|| \ll_\eta 1$.

Combining (5.11) and Lemma 3.4 we get that $|\Delta_{x_{\mathbf{m}}}| \leq |\Delta_\Phi| \ll \xi_\Phi^n \ll (\log ||\mathbf{m}||)^n$ and so choosing ϵ, C appropriately we can make sure that

$||g - I|| \leq Ce^{-|\Delta_x|^\epsilon}$. This concludes the proof that $x_{\mathbf{m}} \in \Omega_M(\epsilon, C)$ and hence the proof of the proposition.

$\square$

References

[AS] M. Aka and U. Shapira, *On the evolution of continued fractions in a fixed quadratic field*. To appear in journal d'analyse. Available arxiv at http://arxiv.org/pdf/1201.1280.pdf.

[Cas52] J. W. S. Cassels, *The product of n inhomogeneous linear forms in n variables*, J. London Math. Soc. **27** (1952), 485–492. MR0050632 (14,358d)

[Dav36] H. Davenport, *Note on a result of siegel*, Acta Arithmetica **2** (1936), 262265.

[ELMV09] M. Einsiedler, E. Lindenstrauss, P. Michel, and A. Venkatesh, *Distribution of periodic torus orbits on homogeneous spaces*, Duke Math. J. **148** (2009), no. 1, 119–174. MR2515103 (2010k:37004)

[HGRS11] R. J. Hans-Gill, M. Raka, and R. Sehmi, *On conjectures of Minkowski and Woods for n = 8*, Acta Arith. **147** (2011), no. 4, 337–385. MR2776094 (2012b:11106)

[Hla50] E. Hlawka, *Über Gitterpunkte in Parallelepipeden*, J. Reine Angew. Math. **187** (1950), 246–252. MR0036783 (12,161d)

[KHGR15] L. Kathuria, R. J. Hans-Gill, and M. Raka, *On a question of Uri Shapira and Barak Weiss*, Indian J. Pure Appl. Math. **46** (2015), no. 3, 287–307. MR3366129

[KPS] A. Kemarsky, F. Paulin, and U. Shapira, *Escape of mass in homogeneous dynamics in positive characteristic*. Preprint.

[McM05] C. T. McMullen, *Minkowski's conjecture, well-rounded lattices and topological dimension*, J. Amer. Math. Soc. **18** (2005), no. 3, 711–734 (electronic). MR2138142 (2006a:11086)

[MS95] S. Mozes and N. Shah, *On the space of ergodic invariant measures of unipotent flows*, Ergodic Theory Dynam. Systems **15** (1995), no. 1, 149–159. MR1314973 (95k:58096)

[Sar07] P. Sarnak, *Reciprocal geodesics*, Analytic number theory, 2007, pp. 217–237. MR2362203 (2009a:11184)

[SW] U. Shapira and B. Weiss, *Stable lattices and the diagonal group*. To appear in Journal of the European Mathematical Society.

[TW03] G. Tomanov and B. Weiss, *Closed orbits for actions of maximal tori on homogeneous spaces*, Duke Math. J. **119** (2003), no. 2, 367–392. MR1997950 (2004g:22006)

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