STABLE LATTICES AND THE DIAGONAL GROUP

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Abstract. Inspired by work of McMullen, we show that any orbit of the diagonal group in the space of lattices accumulates on the set of stable lattices. As consequences, we settle a conjecture of Ramharter concerning the asymptotic behavior of the Mordell constant, and reduce Minkowski's conjecture on products of linear forms to a geometric question, yielding two new proofs of the conjecture in dimensions up to 7.

1. Introduction

Let $n \geq 2$ be an integer, let $G \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{R})$, $\Gamma \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{Z})$, let $A \subset G$ be the subgroup of positive diagonal matrices and let $\mathcal{L}_n \stackrel{\text{def}}{=} G/\Gamma$ be the space of unimodular lattices in $\mathbb{R}^n$. The purpose of this paper is to present a dynamical result regarding the action of A on $\mathcal{L}_n$, and to present some consequences in the geometry of numbers.

A lattice $x \in \mathcal{L}_n$ is called *stable* if for any subgroup $\Lambda \subset x$, the covolume of Λ in $\mathrm{span}(\Lambda)$ is at least 1. In particular the length of the shortest nonzero vector in x is at least 1. Stable lattices have also been called 'semistable', they were introduced in a broad algebro-geometric context by Harder, Narasimhan and Stuhler [13], [8], and were used to develop a reduction theory for the study of the topology of locally symmetric spaces. See Grayson [5] for a clear exposition.

Theorem 1.1. *For any $x \in \mathcal{L}_n$, the orbit-closure $\overline{Ax}$ contains a stable lattice.*

Theorem 1.1 is inspired by a breakthrough result of McMullen [9]. Recall that a lattice in $\mathcal{L}_n$ is called *well-rounded* if its shortest nonzero vectors span $\mathbb{R}^n$. In connection with his work on Minkowski's conjecture, McMullen showed that the closure of any bounded A-orbit in $\mathcal{L}_n$ contains a well-rounded lattice. The set of well-rounded lattices neither contains, nor is contained in, the set of stable lattices; while the set of well-rounded lattices has no interior, the set of stable lattices does, and in fact it occupies all but an exponentially small volume of $\mathcal{L}_n$ for large n. Our proof of Theorem 1.1 closely follows McMullen's. Note however that we do not assume that Ax is bounded.

We apply Theorem 1.1 to two problems in the geometry of numbers. Let $x \in \mathcal{L}_n$ be a unimodular lattice. By a *symmetric box* in $\mathbb{R}^n$ we mean a set of the form $[-a_1, a_1] \times \cdots \times [-a_n, a_n]$, and we say that a symmetric box is *admissible* for x if it contains no nonzero points of x in its interior. The *Mordell constant* of x is defined to be

$$\kappa(x) \stackrel{\text{def}}{=} \frac{1}{2^n} \sup_{\mathcal{B}} \mathrm{Vol}(\mathcal{B}), \tag{1.1}$$

where the supremum is taken over admissible symmetric boxes $\mathcal{B}$, and where $\mathrm{Vol}(\mathcal{B})$ denotes the volume of $\mathcal{B}$. We also write

$$\kappa_n \stackrel{\text{def}}{=} \inf\{\kappa(x) : x \in \mathcal{L}_n\}. \tag{1.2}$$

The infimum in this definition is in fact a minimum, and, as with many problems in the geometry of numbers it is of interest to compute the constants κ_n and identify the lattices realizing the minimum. However this appears to be a very difficult problem, which so far has only been solved for $n = 2, 3$, the latter in a difficult paper of Ramharter [11]. It is also of interest to provide bounds on the asymptotics of κ_n, and in [10], Ramharter conjectured that $\limsup_{n\to\infty} \kappa_n^{1/n \log n} > 0$. As a simple corollary of Theorem 1.1, we validate Ramharter's conjecture, with an explicit bound:

Corollary 1.2. *For all $n \geq 2$,*

$$\kappa_n \geq n^{-n/2}. \tag{1.3}$$

In particular

$$\kappa_n^{1/n \log n} \geq n^{-1/2 \log n} \longrightarrow_{n\to\infty} \frac{1}{\sqrt{e}}.$$

We remark that Corollary 1.2 could also be derived from McMullen's results and a theorem of Birch and Swinnerton-Dyer. We refer the reader to [15] for more information on the possible values of $\kappa(x), x \in \mathcal{L}_n$, and to the preprint [14, §4] for slight improvements.

Our second application concerns Minkowski's conjecture[1], which posits that for any unimodular lattice x, one has

$$\sup_{u \in \mathbb{R}^n} \inf_{v \in x} |N(u - v)| \leq \frac{1}{2^n}, \tag{1.4}$$

where $N(u_1, \ldots, u_d) \stackrel{\text{def}}{=} \prod_j u_j$. Minkowski solved the question for $n = 2$ and several authors resolved the cases $n \leq 5$. In [9], McMullen settled the case $n = 6$. In fact, using his theorem on the A-action on $\mathcal{L}_n$, McMullen showed that in arbitrary dimension n, Minkowski's

[1]It is not clear to us whether Minkowski actually made this conjecture.

conjecture is implied by the statement that any well-rounded lattice $x \subset \mathbb{R}^d$ with $d \leq n$ satisfies

$$\operatorname{covrad}(x) \leq \frac{\sqrt{d}}{2}, \tag{1.5}$$

where $\operatorname{covrad}(x) \stackrel{\text{def}}{=} \max_{u \in \mathbb{R}^d} \min_{v \in x} \|u - v\|$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. At the time of writing [9], (1.5) was known to hold for well-rounded lattices in dimension at most 6, and in recent work of Hans-Gill, Raka, Sehmi and Leetika [6, 7, 12], (1.5) has been proved for well-rounded lattices in dimensions $n = 7, 8, 9$, thus settling Minkowski's question in those cases.

Our work gives two new approaches to Minkowski's conjecture, and each of these approaches yields a new proof of the conjecture in dimensions $n \leq 7$. A direct application of Theorem 1.1 (see Corollary 5.1) shows that it follows in dimension n, from the assertion that for any stable $x \in \mathcal{L}_n$, (1.5) holds. Note that we do not require (1.5) in dimensions less than n. Using the strategy of Woods and Hans-Gill et al, in Theorem 5.8 we define a compact subset $\mathrm{KZS} \subset \mathbb{R}^n$ and a collection of 2^{n-1} subsets $\{\mathcal{W}(\mathcal{I})\}$ of $\mathbb{R}^n$. We show that the assertion $\mathrm{KZS} \subset \bigcup_{\mathcal{I}} \mathcal{W}(\mathcal{I})$ implies Minkowski's conjecture in dimension n. This provides a computational approach to Minkowski's conjecture.

Secondly, an induction using the naturality of stable lattices, leads to the following sufficient condition:

Corollary 1.3. *Suppose that for some dimension n, for all $d \leq n$, any stable lattice $x \in \mathcal{L}_d$ which is a local maximum of the function covrad, satisfies (1.5). Then (1.4) holds for any $x \in \mathcal{L}_n$.*

The local maxima of the function covrad have been studied in depth in recent work of Dutour-Sikirić, Schürmann and Vallentin [3], who characterized them and showed that there are finitely many in each dimension. Dutour-Sikirić has formulated a Conjecture as to which of these have the largest covering radius (see Conjecture 5.9), and has verified his conjecture computationally in dimensions $n \leq 7$. Our results imply that Minkowski's conjecture is a consequence of Conjecture 5.9.

1.1. **Acknowledgements.** Our work was inspired by Curt McMullen's breakthrough paper [9] and many of our arguments are adaptations of arguments appearing in [9]. We are also grateful to Curt McMullen for additional insightful remarks, and in particular for the suggestion to study the set of stable lattices in connection with the A-action on

$\mathcal{L}_n$. We also thank Mathieu Dutour-Sikirić, Rajinder Hans-Gill, Günter Harder, Gregory Minton and Gerhard Ramharter for useful discussions.

We are grateful to the referee for helping us improve the presentation of our results. A previous version of this paper, which included several other results, was circulated under the title 'On stable lattices and the diagonal group.' At the referee's suggestion, the current version presents our main results but omits others. For the original version the reader is referred to [14]. Additional results will appear elsewhere.

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2. Orbit closures and stable lattices

Given a lattice $x \in \mathcal{L}_n$ and a subgroup $\Lambda \subset x$, we denote by $r(\Lambda)$ the rank of Λ and by $|\Lambda|$ the covolume of Λ in the linear subspace $\mathrm{span}(\Lambda)$. Let

$$\mathcal{V}(x) \stackrel{\mathrm{def}}{=} \left\{ |\Lambda|^{\frac{1}{r(\Lambda)}} : \Lambda \subset x \right\},$$
$$\alpha(x) \stackrel{\mathrm{def}}{=} \min \mathcal{V}(x). \tag{2.1}$$

Since we may take $\Lambda = x$ we have $\alpha(x) \leq 1$ for all $x \in \mathcal{L}_n$, and x is stable precisely if $\alpha(x) = 1$. Observe that $\mathcal{V}(x)$ is a countable discrete subset of the positive reals, and hence the minimum in (2.1) is attained. Also note that the function α is a variant of the 'length of the shortest vector'; it is continuous and the sets $\{x : \alpha(x) \geq \varepsilon\}$ are an exhaustion of $\mathcal{L}_n$ by compact sets.

We begin by explaining the strategy for proving Theorem 1.1, which is identical to the one used by McMullen. For a lattice $x \in X$ and $\varepsilon > 0$ we define an open cover $\mathcal{U}^{x,\varepsilon} = \{U_k^{x,\varepsilon}\}_{k=1}^n$ of the diagonal group A, where if $a \in U_k^{x,\varepsilon}$ then $\alpha(ax)$ is 'almost attained' by a subgroup of rank k. In particular, if $a \in U_n^{x,\varepsilon}$ then ax is 'almost stable'. The main point is to show that for any $\varepsilon > 0$, $U_n^{x,\varepsilon} \neq \varnothing$; for then, taking $\varepsilon_j \to 0$ and $a_j \in A$ such that $a_j \in U_n^{x,\varepsilon_j}$, we find (passing to a subsequence) that $a_j x$ converges to a stable lattice.

In order to establish that $U_n^{x,\varepsilon} \neq \varnothing$, we apply a topological result of McMullen (Theorem 3.3) regarding open covers which is reminiscent of the classical result of Lebesgue that asserts that in an open cover of Euclidean n-space by bounded balls there must be a point which is covered $n+1$ times. We will work to show that the cover $\mathcal{U}^{x,\varepsilon}$ satisfies the assumptions of Theorem 3.3. We will be able to verify these assumptions when the orbit Ax is bounded. In §2.1 we reduce the proof of Theorem 1.1 to this case.

2.1. **Reduction to bounded orbits.** Using a result of Birch and Swinnerton-Dyer, we will now show that it suffices to prove Theorem 1.1 under the assumption that the orbit $Ax \subset \mathcal{L}_n$ is bounded; that is, that $\overline{Ax}$ is compact. In this subsection we will denote A, G by A_n, G_n as various dimensions will appear.

For a matrix $g \in G_n$ we denote by $[g] \in \mathcal{L}_n$ the corresponding lattice. If

$$g = \begin{pmatrix} g_1 & * & \dots & * \\ 0 & g_2 & \dots & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & g_k \end{pmatrix} \tag{2.2}$$

where $g_i \in G_{n_i}$ for each i, then we say that g is in *upper triangular block form* and refer to the g_i's as the *diagonal blocks*. Note that in this definition, we insist that each g_i is of determinant one.

Lemma 2.1. *Let $x = [g] \in \mathcal{L}_n$ where g is in upper triangular block form as in (2.2) and for each $1 \le i \le k$, $[g_i]$ is a stable lattice in $\mathcal{L}_{n_i}$. Then x is stable.*

Proof. By induction, in proving the Lemma we may assume that $k = 2$. Let us denote the standard basis of $\mathbb{R}^n$ by $\mathbf{e}_1, \dots, \mathbf{e}_n$, let us write $n = n_1 + n_2$, $V_1 \stackrel{\text{def}}{=} \operatorname{span}\{\mathbf{e}_1, \dots, \mathbf{e}_{n_1}\}$, $V_2 \stackrel{\text{def}}{=} \operatorname{span}\{\mathbf{e}_{n_1+1} \dots, \mathbf{e}_n\}$, and let $\pi : \mathbb{R}^n \to V_2$ be the natural projection. By construction we have $x \cap V_1 = [g_1], \pi(x) = [g_2]$.

Let $\Lambda \subset x$ be a subgroup, write $\Lambda_1 \stackrel{\text{def}}{=} \Lambda \cap V_1$ and choose a direct complement $\Lambda_2 \subset \Lambda$, that is

$$\Lambda = \Lambda_1 + \Lambda_2, \quad \Lambda_1 \cap \Lambda_2 = \{0\}.$$

We claim that

$$|\Lambda| = |\Lambda_1| \cdot |\pi(\Lambda_2)|. \tag{2.3}$$

To see this we recall that one may compute $|\Lambda|$ via the Gram-Schmidt process. Namely, one begins with a set of generators v_j of Λ and successively defines $u_1 = v_1$ and u_j is the orthogonal projection of v_j on $\operatorname{span}(v_1, \dots, v_{j-1})^\perp$. In these terms, $|\Lambda| = \prod_j \|u_j\|$. Since π is an orthogonal projection and $\Lambda \cap V_1$ is in $\ker \pi$, (2.3) is clear from the above description.

The discrete subgroup Λ_1, when viewed as a subgroup of $[g_1] \in \mathcal{L}_{n_1}$ satisfies $|\Lambda_1| \ge 1$ because $[g_1]$ is assumed to be stable. Similarly $\pi(\Lambda_2) \subset [g_2] \in \mathcal{L}_{n_2}$ satisfies $|\pi(\Lambda_2)| \ge 1$, hence $|\Lambda| \ge 1$. $\square$

Lemma 2.2. *Let $x \in \mathcal{L}_n$ and assume that $\overline{Ax}$ contains a lattice $[g]$ with g of upper triangular block form as in (2.2). For each $1 \le i \le k$,* *suppose* $[h_i] \in \overline{A_{n_i}[g_i]} \subset \mathcal{L}_{n_i}$*. Then there exists a lattice* $[h] \in \overline{Ax}$ *such that* h *has the form* (2.2) *with* h_i *as its diagonal blocks.*

Proof. Let Ω be the set of all lattices $[g]$ of a fixed triangular form as in (2.2). Then Ω is a closed subset of $\mathcal{L}_n$ and there is a projection

$$\tau : \Omega \to \mathcal{L}_{n_1} \times \cdots \times \mathcal{L}_{n_k}, \quad \tau([g]) = ([g_1], \ldots, [g_k]).$$

The map τ has a compact fiber and is equivariant with respect to the action of $\widetilde{A} \overset{\text{def}}{=} A_{n_1} \times \cdots \times A_{n_k}$. By assumption, there is a sequence $\tilde{a}_j = \left(a_1^{(j)}, \ldots, a_k^{(j)}\right)$, $a_i^{(j)} \in A_{n_i}$ in $\widetilde{A}$ such that $a_i^{(j)}[g_i] \to [h_i]$, then after passing to a subsequence, $\tilde{a}_j[g] \to [h]$ where h has the required properties. Since $\overline{Ax} \supset \overline{\widetilde{A}[g]}$, the claim follows. ☐

Lemma 2.3. *Let* $x \in \mathcal{L}_n$*. Then there is* $[g] \in \overline{Ax}$ *such that, up to a possible permutation of the coordinates,* g *is of upper triangular block form as in* (2.2) *and each* $A_{n_i}[g_i] \subset \mathcal{L}_{n_i}$ *is bounded.*

Proof. If the orbit Ax is bounded there is nothing to prove. According to Birch and Swinnerton-Dyer [1], if Ax is unbounded then $\overline{Ax}$ contains a lattice with a representative as in (2.2) (up to a possible permutation of the coordinates) with $k = 2$. Now the claim follows using induction and appealing to Lemma 2.2. ☐

Proposition 2.4. It is enough to establish Theorem 1.1 for lattices having a bounded A-orbit.

Proof. Let $x \in \mathcal{L}_n$ be arbitrary. By Lemma 2.3, $\overline{Ax}$ contains a lattice $[g]$ with g of upper triangular block form (up to a possible permutation of the coordinates) with diagonal blocks representing lattices with bounded orbits under the corresponding diagonal groups. Assuming Theorem 1.1 for lattices having bounded orbits, and applying Lemma 2.2 we may take g whose diagonal blocks represent stable lattices. By Lemma 2.1, $[g]$ is stable as well. ☐

2.2. **Some technical preparations.** We now discuss the subgroups of a lattice $x \in \mathcal{L}_n$ which almost attain the minimum $\alpha(x)$ in (2.1).

Definition 2.5. Given a lattice $x \in \mathcal{L}_n$ and $\delta > 0$, let

$$\begin{aligned}
\mathrm{Min}_\delta(x) &\overset{\text{def}}{=} \left\{ \Lambda \subset x : |\Lambda|^{\frac{1}{r(\Lambda)}} < (1+\delta)\alpha(x) \right\}, \\
\mathbf{V}_\delta(x) &\overset{\text{def}}{=} \mathrm{span}\left(\bigcup \{\Lambda : \Lambda \in \mathrm{Min}_\delta(x)\} \right), \\
\dim_\delta(x) &\overset{\text{def}}{=} \dim \mathbf{V}_\delta(x).
\end{aligned}$$

We will need the following technical statement.

Lemma 2.6. *For any $\rho > 0$ there exists a neighborhood of the identity $W \subset G$ with the following property. Suppose $2\rho \le \delta_0 \le d+1$ and suppose $x \in \mathcal{L}_n$ is such that $\dim_{\delta_0-\rho}(x) = \dim_{\delta_0+\rho}(x)$. Then for any $g \in W$ and any $\delta \in \left(\delta_0 - \frac{\rho}{2}, \delta_0 + \frac{\rho}{2}\right)$ we have*

$$\boldsymbol{V}_\delta(gx) = g\,\boldsymbol{V}_{\delta_0}(x). \tag{2.4}$$

In particular, there is $1 \le k \le n$ such that for any $g \in W$ and any $\delta \in \left(\delta_0 - \frac{\rho}{2}, \delta_0 + \frac{\rho}{2}\right)$, $\dim_\delta(gx) = k$.

Proof. Let $c > 1$ be chosen close enough to 1 so that for $2\rho \le \delta_0 \le d+1$ we have

$$c^2\left(1 + \delta_0 + \frac{\rho}{2}\right) < 1 + \delta_0 + \rho \quad \text{and} \quad \frac{1+\delta_0 - \frac{\rho}{2}}{c^2} > 1 + \delta_0 - \rho. \tag{2.5}$$

Let W be a small enough neighborhood of the identity in G, so that for any discrete subgroup $\Lambda \subset \mathbb{R}^n$ we have

$$g \in W \quad \Longrightarrow \quad c^{-1}\,|\Lambda|^{\frac{1}{r(\Lambda)}} \le |g\Lambda|^{\frac{1}{r(g\Lambda)}} \le c\,|\Lambda|^{\frac{1}{r(\Lambda)}}. \tag{2.6}$$

Such a neighborhood exists since the linear action of G on $\bigoplus_{k=1}^n \bigwedge_1^k \mathbb{R}^n$ is continuous, and since we can write $|\Lambda| = \|v_1 \wedge \cdots \wedge v_r\|$ where $v_1, \ldots, v_r$ is a generating set for Λ. It follows from (2.6) that for any $x \in \mathcal{L}_n$ and $g \in W$ we have

$$c^{-1}\alpha(x) \le \alpha(gx) \le c\alpha(x). \tag{2.7}$$

Let $\delta \in \left(\delta_0 - \frac{\rho}{2}, \delta_0 + \frac{\rho}{2}\right)$ and $g \in W$. We will show below that

$$g\,\mathrm{Min}_{\delta_0-\rho}(x) \subset \mathrm{Min}_\delta(gx) \subset g\,\mathrm{Min}_{\delta_0+\rho}(x). \tag{2.8}$$

Note first that (2.8) implies the assertion of the Lemma; indeed, since $\mathbf{V}_{\delta_1}(x) \subset \mathbf{V}_{\delta_2}(x)$ for $\delta_1 < \delta_2$, and since we assumed that $\dim_{\delta_0-\rho}(x) = \dim_{\delta_0+\rho}(x)$, we see that $\mathbf{V}_{\delta_0}(x) = \mathbf{V}_\delta(x)$ for $\delta_0 - \rho \le \delta \le \delta_0 + \rho$. So by (2.5), the subspaces spanned by the two sides of (2.8) are equal to $g\mathbf{V}_{\delta_0}(x)$ and (2.4) follows.

It remains to prove (2.8). Let $\Lambda \in \mathrm{Min}_{\delta_0-\rho}(x)$. Then we find

$$\begin{aligned} |g\Lambda|^{\frac{1}{r(g\Lambda)}} &\overset{(2.6)}{\le} c\,|\Lambda|^{\frac{1}{r(\Lambda)}} \le c(1+\delta_0-\rho)\alpha(x) \\ &\overset{(2.5)}{\le} c^{-1}\left(1+\delta_0 - \frac{\rho}{2}\right)\alpha(x) \overset{(2.7)}{<} (1+\delta)\alpha(gx). \end{aligned}$$

By definition this means that $g\Lambda \in \mathrm{Min}_\delta(gx)$ which establishes the first inclusion in (2.8). The second inclusion is similar and is left to the reader. $\square$

2.3. **The cover of A.** Let $x \in \mathcal{L}_n$ and let $\varepsilon > 0$ be given. Define $\mathcal{U}^{x,\varepsilon} = \{U_i^{x,\varepsilon}\}_{i=1}^n$ where

$$U_k^{x,\varepsilon} \stackrel{\text{def}}{=} \{a \in A : \dim_\delta(ax) = k \text{ for } \delta \text{ in a neighborhood of } k\varepsilon\}. \quad (2.9)$$

Theorem 2.7. *Let $x \in \mathcal{L}_n$ be such that Ax is bounded. Then for any $\varepsilon \in (0,1)$, $U_n^{x,\varepsilon} \neq \varnothing$.*

In this subsection we will reduce the proof of Theorem 1.1 to Theorem 2.7. This will be done via the following statement, which could be interpreted as saying that a lattice satisfying $\dim_\delta(x) = n$ is 'almost stable'.

Lemma 2.8. *For each n, there exists a positive function $\psi(\delta)$ with $\psi(\delta) \to_{\delta\to 0} 0$, such that for any $x \in \mathcal{L}_n$,*

$$\{\Lambda_i\}_{i=1}^{\ell} \subset \mathrm{Min}_\delta(x) \implies \Lambda_1 + \cdots + \Lambda_\ell \in \mathrm{Min}_{\psi(\delta)}(x). \quad (2.10)$$

In particular, if $\dim_\delta(x) = n$ then $\alpha(x) \geq (1+\psi(\delta))^{-1}$.

Proof. Let Λ, Λ' be two discrete subgroups of $\mathbb{R}^d$. The following inequality is straightforward to prove via the Gram-Schmidt procedure for computing $|\Lambda|$:

$$|\Lambda + \Lambda'| \leq \frac{|\Lambda| \cdot |\Lambda'|}{|\Lambda \cap \Lambda'|}. \quad (2.11)$$

Here we adopt the convention that $|\Lambda \cap \Lambda'| = 1$ when $\Lambda \cap \Lambda' = \{0\}$. By induction on $\ell \leq n$, we now prove the existence of a function $\psi_\ell(\delta) \stackrel{\delta\to 0}{\longrightarrow} 0$ such that for any $x \in \mathcal{L}_n$ and any $\{\Lambda_i\}_{i=1}^{\ell} \subset \mathrm{Min}_\delta(x)$, we have $\Lambda_1 + \cdots + \Lambda_\ell \in \mathrm{Min}_{\psi_\ell(\delta)}(x)$. For $\ell = 1$ one can trivially pick $\psi_1(\delta) = \delta$. Assuming the existence of $\psi_{\ell-1}$, set

$$\psi_\ell(\delta) \stackrel{\text{def}}{=} \max \left((1+\delta)^{r(\Lambda)} (1 + \psi_{\ell-1}(\delta))^{r(\Lambda')} \right)^{\frac{1}{r(\Lambda+\Lambda')}} - 1,$$

where the maximum is taken over all possible values of $r(\Lambda), r(\Lambda'), r(\Lambda+\Lambda')$. Clearly $\psi_\ell(\delta) \longrightarrow_{\delta\to 0} 0$, and given $x \in \mathcal{L}_n$ and $\Lambda_1, \ldots, \Lambda_\ell \in \mathrm{Min}_\delta(x)$, set $\Lambda = \Lambda_1$, $\Lambda' = \Lambda_2 + \cdots + \Lambda_\ell$, $\alpha = \alpha(x)$ and note that $r(\Lambda + \Lambda') = r(\Lambda) + r(\Lambda') - r(\Lambda \cap \Lambda')$. We deduce from (2.11) and the definitions that

$$\begin{aligned} |\Lambda + \Lambda'| \leq \frac{|\Lambda| \cdot |\Lambda'|}{|\Lambda \cap \Lambda'|} &\leq \frac{((1+\delta)\alpha)^{r(\Lambda)} \left((1+\psi_{\ell-1}(\delta))\alpha\right)^{r(\Lambda')}}{\alpha^{r(\Lambda\cap\Lambda')}} \\ &= (1+\delta)^{r(\Lambda)} (1+\psi_{\ell-1}(\delta))^{r(\Lambda')} \alpha^{r(\Lambda+\Lambda')}, \end{aligned}$$

and so $\Lambda + \Lambda' \in \mathrm{Min}_{\psi_\ell(\delta)}(x)$ as desired. This completes the inductive step.

We take $\psi(\delta) \stackrel{\mathrm{def}}{=} \max_{\ell=1}^{n} \psi_\ell(\delta)$. If $\ell \leq n$ then (2.10) holds by construction. If $\ell > n$ one can find a subsequence $1 \leq i_1 < i_2 \cdots < i_d \leq n$ such that $r(\sum_{i=1}^{\ell} \Lambda_i) = r(\sum_{j=1}^{d} \Lambda_{i_j})$ and in particular, $\sum_{j=1}^{d} \Lambda_{i_j}$ is of finite index in $\sum_{i=1}^{\ell} \Lambda_i$. From the first part of the argument we see that $\sum_{j=1}^{d} \Lambda_{i_j} \in \mathrm{Min}_{\psi(\delta)}(x)$ and as the covolume of $\sum_{i=1}^{\ell} \Lambda_i$ is not larger than that of $\sum_{j=1}^{d} \Lambda_{i_j}$ we deduce that $\sum_{i=1}^{\ell} \Lambda_i \in \mathrm{Min}_{\psi_\ell(\delta)}(x)$ as well.

To verify the last assertion, note that when $\dim_\delta(x) = n$, (2.10) implies the existence of a finite index subgroup x' of x belonging to $\mathrm{Min}_{\psi(\delta)}(x)$. In particular, $1 \leq |x'|^{\frac{1}{n}} \leq (1+\psi(\delta))\alpha(x)$ as desired. □

Proof of Theorem 1.1 assuming Theorem 2.7. By Proposition 2.4 we may assume that Ax is bounded. Let $\varepsilon_j \in (0,1)$ so that $\varepsilon_j \to_j 0$. By Theorem 2.7 we know that $U_n^{x,\varepsilon_j} \neq \varnothing$. This means there is a sequence $a_j \in A$ such that $\dim_{\delta_j}(a_j x) = n$ where $\delta_j = n\varepsilon_j \to 0$. The sequence $\{a_j x\}$ is bounded, and hence has limit points, so passing to a subsequence we let $x' \stackrel{\mathrm{def}}{=} \lim a_j x$. By Lemma 2.8 we have

$$1 \geq \limsup_j \alpha(a_j x) \geq \liminf_j \alpha(a_j x) \geq \lim_j (1+\psi(\delta_j))^{-1} = 1,$$

which shows that $\lim_j \alpha(a_j x) = 1$. The function α is continuous on $\mathcal{L}_n$ and therefore $\alpha(x') = 1$, i.e. $x' \in \overline{Ax}$ is stable. □

3. Covers of Euclidean space

In this section we will prove Theorem 2.7, thus completing the proof of Theorem 1.1. Our main tool will be McMullen's Theorem 3.3. Before stating it we introduce some terminology. We fix an invariant metric on A, and let $R > 0$ and $k \in \{0, \ldots, n-1\}$.

Definition 3.1. We say that a subset $U \subset A$ is (R,k)-*almost affine* if it is contained in an R-neighborhood of a coset of a connected k-dimensional subgroup of A.

Definition 3.2. An open cover $\mathcal{U}$ of A is said to have *inradius* $r > 0$ if for any $a \in A$ there exists $U \in \mathcal{U}$ such that $B_r(a) \subset U$, where $B_r(a)$ denotes the ball in A of radius r around a.

Theorem 3.3 (Theorem 5.1 of [9])**.** *Let $\mathcal{U}$ be an open cover of A with inradius $r > 0$ and let $R > 0$. Suppose that for any $1 \leq k \leq n-1$, every connected component V of the intersection of k distinct elements of $\mathcal{U}$ is $(R,(n-1-k))$-almost affine. Then there is a point in A which belongs to at least n distinct elements of $\mathcal{U}$. In particular, there are at least n distinct non-empty sets in $\mathcal{U}$.*

3.1. **Verifying the hypotheses of Theorem 3.3.** Below we fix a compact set $K \subset \mathcal{L}_n$ and a lattice x for which $Ax \subset K$. Furthermore, we fix $\varepsilon > 0$ and denote the collection $\mathcal{U}^{x,\varepsilon}$ defined in (2.9) by $\mathcal{U} = \{U_i\}_{i=1}^n$.

Lemma 3.4. *The collection $\mathcal{U}$ forms an open cover of A with positive inradius.*

Proof. The fact that the sets $U_i \subset A$ are open follows readily from the requirement in (2.9) that $\dim_\delta$ is constant for δ in a neighborhood of $k\varepsilon$. Given $a \in A$, let $1 \le k_0 \le n$ be the minimal number k for which $\dim_{(k+\frac{1}{2})\varepsilon}(ax) \le k$ (this inequality holds trivially for $k = n$). From the minimality of k_0 we conclude that $\dim_\delta(ax) = k_0$ for any $\delta \in \left[\left(k_0 - \frac{1}{2}\right)\varepsilon, \left(k_0 + \frac{1}{2}\right)\varepsilon\right]$. This shows that $a \in U_{k_0}$ so $\mathcal{U}$ is indeed a cover of A.

We now show that the cover has positive inradius. Let $W \subset G$ be the open neighborhood of the identity obtained from Lemma 2.6 for $\rho \overset{\text{def}}{=} \frac{\varepsilon}{2}$. Taking $\delta_0 \overset{\text{def}}{=} k_0\varepsilon$ we find that for any $g \in W$, $\delta \in \left(\left(k_0 - \frac{1}{4}\right)\varepsilon, \left(k_0 + \frac{1}{4}\right)\varepsilon\right)$ we have that $\dim_\delta(gax) = k_0$. This shows that $(W \cap A)a \subset U_{k_0}$. Since $W \cap A$ is an open neighborhood of the identity in A and the metric on A is invariant under translation by elements of A, there exists $r > 0$ (independent of k_0 and a) so that $B_r(a) \subset U_{k_0}$. In other words, the inradius of $\mathcal{U}$ is positive as desired. $\square$

The following will be used for verifying the second hypothesis of Theorem 3.3.

Lemma 3.5. *There exists $R > 0$ such that any connected component of U_k is $(R, k-1)$-almost affine.*

Definition 3.6. For a discrete subgroup $\Lambda \subset \mathbb{R}^d$ of rank k, let

$$c(\Lambda) \overset{\text{def}}{=} \inf \left\{ |a\Lambda|^{1/k} : a \in A \right\},$$

and say that Λ is *incompressible* if $c(\Lambda) > 0$.

Lemma 3.5 follows from:

Theorem 3.7 ([9, Theorem 6.1])**.** *For any positive c, C there exists $R > 0$ such that if $\Lambda \subset \mathbb{R}^n$ is an incompressible discrete subgroup of rank k with $c(\Lambda) \ge c$ then $\left\{a \in A : |a\Lambda|^{1/k} \le C\right\}$ is (R, j)-almost affine for some $j \le \gcd(k, n) - 1$.*

Proof of Lemma 3.5. We first claim that there exists $c > 0$ such that for any discrete subgroup $\Lambda \subset x$ we have that $c(\Lambda) \ge c$. To see this, recall that Ax is contained in a compact subset K, and hence by Mahler's compactness criterion, there is a positive lower bound on the length of any non-zero vector belonging to a lattice in K. On the other hand, Minkowski's convex body theorem shows that the shortest nonzero vector in a discrete subgroup $\Lambda \subset \mathbb{R}^n$ is bounded above by a constant multiple of $|\Lambda|^{1/r(\Lambda)}$. This implies the claim.

In light of Theorem 3.7, it suffices to show that there is $C > 0$ such that if $V \subset U_k$ is a connected component, then there exists $\Lambda \subset x$ such that $V \subset \left\{ a \in A : |a\Lambda|^{1/k} \leq C \right\}$. For any $1 \leq k \leq n$, write $\mathbf{gr}_k$ for the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. Define

$$\mathcal{M} : U_k \to \mathbf{gr}_k, \quad \mathcal{M}(a) \stackrel{\text{def}}{=} a^{-1}\mathbf{V}_{k\varepsilon}(ax).$$

Observe that $\mathcal{M}$ is locally constant on U_k. Indeed, by definition of U_k, for $a_0 \in U_k$ there exists $0 < \rho < \frac{\varepsilon}{2}$ such that $\dim_\delta(a_0x) = k$ for any $\delta \in (k\varepsilon - \rho, k\varepsilon + \rho)$. Applying Lemma 2.6 for the lattice a_0x with ρ and $\delta_0 = k\varepsilon$ we see that for any a in a neighborhood of the identity in A,

$$\mathcal{M}(aa_0) = a_0^{-1}a^{-1}\mathbf{V}_{k\varepsilon}(aa_0x) = a_0^{-1}\mathbf{V}_{k\varepsilon}(a_0x) = \mathcal{M}(a_0).$$

Now let $\Lambda \stackrel{\text{def}}{=} x \cap \mathcal{M}(a)$ where $a \in V$; Λ is well-defined since $\mathcal{M}$ is locally constant. Then for $a \in V$,

$$a\Lambda = a(x \cap \mathcal{M}(a)) = a(x \cap a^{-1}\mathbf{V}_{k\varepsilon}(ax)) = ax \cap \mathbf{V}_{k\varepsilon}(ax).$$

By Lemma 2.8 we have that

$$|a\Lambda|^{1/k} = |ax \cap \mathbf{V}_{k\varepsilon}(ax)|^{1/k} < (1 + \psi(k\varepsilon))\alpha(ax).$$

Since $\alpha(ax) \leq 1$ we may take $C \stackrel{\text{def}}{=} 1 + \psi(k\varepsilon)$ to complete the proof. $\square$

Proof of Theorem 2.7. Assume by contradiction that Ax is bounded but $U_n^{x,\varepsilon} = \varnothing$ for some $\varepsilon \in (0,1)$. Then by Lemma 3.4,

$$\mathcal{U} \stackrel{\text{def}}{=} \{U_1, \ldots, U_{n-1}\}, \text{ where } U_j \stackrel{\text{def}}{=} U_j^{x,\varepsilon},$$

is a cover of A of positive inradius. Moreover, if V is a connected component of $U_{j_1} \cap \cdots \cap U_{j_k}$ with $j_1 < \cdots < j_k \leq n-1$, then $V_k \subset U_{j_1}$ and $j_1 \leq n-k$. So in light of Lemma 3.5, the hypotheses of Theorem 3.3 are satisfied. We deduce that $\mathcal{U} = \{U_1, \ldots, U_{n-1}\}$ contains at least n elements, which is impossible. $\square$

4. Bounds on Mordell's constant

In analogy with (2.1) we define for any $x \in \mathcal{L}_n$ and $1 \leq k \leq n$,

$$\mathcal{V}_k(x) \stackrel{\text{def}}{=} \left\{ |\Lambda|^{1/r(\Lambda)} : \Lambda \subset x, r(\Lambda) = k \right\}, \tag{4.1}$$

$$\alpha_k(x) \stackrel{\text{def}}{=} \min \mathcal{V}_k(x). \tag{4.2}$$

The following is clearly a consequence of Theorem 1.1:

Corollary 4.1. *For any $x \in \mathcal{L}_n$, any $\varepsilon > 0$ and any $k \in \{1, \dots, n\}$ there is $a \in A$ such that $\alpha_k(ax) \geq 1 - \varepsilon$.*

As the lattice $x = \mathbb{Z}^n$ shows, the constant 1 appearing in this corollary cannot be improved for any k. Note also that the case $k = 1$ of Corollary 4.1, although not stated explicitly in [9], could be derived easily from McMullen's results in conjunction with [1].

Proof of Corollary 1.2. Since the A-action maps a symmetric box $\mathcal{B}$ to a symmetric box of the same volume, the function $\kappa : \mathcal{L}_n \to \mathbb{R}$ in (1.1) is A-invariant. By the case $k = 1$ of Corollary 4.1, for any $\varepsilon > 0$ and any $x \in \mathcal{L}_n$ there is $a \in A$ such that ax does not contain nonzero vectors of Euclidean length at most $1 - \varepsilon$, and hence does not contain nonzero vectors in the cube $\left[-\left(\frac{1}{\sqrt{n}} - \varepsilon\right), \left(\frac{1}{\sqrt{n}} - \varepsilon\right)\right]^n$. This implies that $\kappa(x) \geq \left(\frac{1}{\sqrt{n}}\right)^n$, as claimed. □

The bound (1.3) is not tight for any n. This is shown in [14], along with several slight improvements of (1.3). For example we prove that if $n \geq 5$ is congruent to 1 mod 4, then

$$\kappa_n \geq \frac{1}{\sqrt{2n-1}(n-1)^{(n-1)/2}}.$$

Similar slight improvements can be obtained for all n not divisible by 4. See [14] for more details.

5. Two strategies for Minkowski's conjecture

We begin by recalling the well-known Davenport-Remak strategy for proving Minkowski's conjecture. The function $N(u) = \prod_1^n u_i$ is clearly A-invariant, and it follows that the quantity

$$\widetilde{N}(x) \stackrel{\text{def}}{=} \sup_{u \in \mathbb{R}^n} \inf_{v \in x} |N(u - v)|$$

appearing in (1.4) is A-invariant. Moreover, it is easy to show that if $x_j \to x$ in $\mathcal{L}_n$ then $\widetilde{N}(x) \geq \limsup_j \widetilde{N}(x_j)$. Therefore, in order to show the estimate (1.4) for $x' \in \mathcal{L}_n$, it is enough to show it for some $x \in \overline{Ax'}$.

Suppose that x satisfies (1.5) with $d = n$; that is for every $u \in \mathbb{R}^n$ there is $v \in x$ such that $\|u - v\| \leq \frac{\sqrt{n}}{2}$. Then applying the inequality of arithmetic and geometric means one finds

$$\prod_1^n \left(|u_i - v_i|^2\right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_1^n |u_i - v_i|^2 \leq \frac{1}{4}$$

which implies $|N(u - v)| \leq \frac{1}{2^n}$. The upshot is that in order to prove Minkowski's conjecture, it is enough to prove that for every $x' \in \mathcal{L}_n$ there is $x \in \overline{Ax}$ satisfying (1.5). So in light of Theorem 1.1 we obtain:

Corollary 5.1. *If all stable lattices in $\mathcal{L}_n$ satisfy (1.5), then Minkowski's conjecture is true in dimension n.*

In the next two subsections, we outline two strategies for establishing that all stable lattices satisfy (1.5). Both strategies yield affirmative answers in dimensions $n \leq 7$, thus providing new proofs of Minkowski's conjecture in these dimensions.

5.1. **Using Korkine-Zolotarev reduction.** Korkine-Zolotarev reduction is a classical method for choosing a basis $v_1, \ldots, v_n$ of a lattice $x \in \mathcal{L}_n$. Namely one takes for v_1 a shortest nonzero vector of x and denotes its length by A_1. Then, proceeding inductively, for v_i one takes a vector whose projection onto $(\mathrm{span}(v_1, \ldots, v_{i-1}))^\perp$ is shortest (among those with nonzero projection), and denotes the length of this projection by A_i. In case there is more than one shortest vector the process is not uniquely defined. Nevertheless we call $A_1, \ldots, A_n$ the *diagonal KZ coefficients of* x (with the understanding that these may be multiply defined for some measure zero subset of $\mathcal{L}_n$). Since x is unimodular we always have

$$\prod A_i = 1. \tag{5.1}$$

Korkine and Zolotarev proved the bounds

$$A_{i+1}^2 \geq \frac{3}{4} A_i^2, \quad A_{i+2}^2 \geq \frac{2}{3} A_i^2. \tag{5.2}$$

A method introduced by Woods and developed further in [6] leads to an upper bound on $\mathrm{covrad}(x)$ in terms of the diagonal KZ coefficients. The method relies on the following estimate. Below $\gamma_n \stackrel{\mathrm{def}}{=} \sup_{x \in \mathcal{L}_n} \alpha_1^2(x)$ (where α_1 is defined via (4.1)) is the *Hermite constant.*

Lemma 5.2 ([16], Lemma 1)**.** *Suppose that x is a lattice in $\mathbb{R}^n$ of covolume d, and suppose that $2A_1^n \geq d\gamma_{n+1}^{(n+1)/2}$. Then*

$$\mathrm{covrad}^2(x) \leq A_1^2 - \frac{A_1^{2n+2}}{d^2 \gamma_{n+1}^{n+1}}.$$

Woods also used the following observation:

Lemma 5.3 ([16], Lemma 2)**.** *Let x be a lattice in $\mathbb{R}^n$, let Λ be a subgroup, and let Λ' denote the projection of x onto* $(\mathrm{span}\Lambda)^{\perp}$*. Then*

$$\mathrm{covrad}^2(x) \le \mathrm{covrad}^2(\Lambda) + \mathrm{covrad}^2(\Lambda')$$

As a consequence of Lemmas 5.2 and 5.3, we obtain:

Proposition 5.4. Suppose $A_1, \ldots, A_n$ are diagonal KZ coefficients of $x \in \mathcal{L}_n$ and suppose $n_1, \ldots, n_k$ are positive integers with $n = n_1 + \cdots + n_k$. Set

$$m_i \stackrel{\mathrm{def}}{=} n_1 + \cdots + n_i \text{ and } d_i \stackrel{\mathrm{def}}{=} \prod_{j=m_{i-1}+1}^{m_i} A_j. \tag{5.3}$$

If

$$2A_{m_{i-1}+1} \ge d_i \gamma_{n_i+1}^{(n_i+1)/2} \tag{5.4}$$

for each i, then

$$\mathrm{covrad}^2(x) \le \sum_{i=1}^{k} \left(A_{m_{i-1}+1}^2 - \frac{A_{m_{i-1}+1}^{2n_i+2}}{d_i^2 \gamma_{n_i+1}^{n_i+1}} \right) \tag{5.5}$$

Proof. Let $v_1, \ldots, v_n$ be the basis of x obtained by the Korkine Zolotarev reduction process. Let Λ_1 be the subgroup of x generated by $v_1, \ldots, v_{n_1}$, and for $i = 2, \ldots, k$ let Λ_i be the projection onto $(\bigoplus_1^{i-1} \Lambda_j)^{\perp}$ of the subgroup of x generated by $v_{m_{i-1}+1}, \ldots, v_{m_i}$. This is a lattice of dimension m_i, and arguing as in the proof of (2.3) we see that it has covolume d_i. The assumption (5.4) says that we may apply Lemma 5.2 to each Λ_i. We obtain

$$\mathrm{covrad}^2(\Lambda_i) \le A_{m_{i-1}+1}^2 - \frac{A_{m_{i-1}+1}^{2n_i+2}}{d_i^2 \gamma_{n_i+1}^{n_i+1}}$$

for each i, and we combine these estimates using Lemma 5.3 and an obvious induction. $\square$

Remark 5.5. Note that it is an open question to determine the numbers γ_n; however, if we have a bound $\tilde{\gamma}_n \ge \gamma_n$ we may substitute it into Proposition 5.4 in place of γ_n, as this only makes the requirement (5.4) stricter and the conclusion (5.5) weaker.

Our goal is to apply this method to the problem of bounding the covering radius of stable lattices. We note:

Proposition 5.6. If x is stable then we have the inequalities

$$A_1 \ge 1, \quad A_1 A_2 \ge 1, \quad \ldots \quad A_1 \cdots A_{n-1} \ge 1. \tag{5.6}$$

Proof. In the above terms, the number $A_1 \cdots A_i$ is equal to $|\Lambda|$ where Λ is the subgroup of x generated by $v_1, \ldots, v_i$. $\square$

This motivates the following:

Definition 5.7. We say that an n-tuple of positive real numbers $A_1, \ldots, A_n$ is *KZ stable* if the inequalities (5.1), (5.2), (5.6) are satisfied. We denote the set of KZ stable n-tuples by KZS.

Note that KZS is a compact subset of $\mathbb{R}^n$. Recall that a *composition of* n is an ordered k-tuple $(n_1, \ldots, n_k)$ of positive integers, such that $n = n_1 + \ldots + n_k$. As an immediate application of Corollary 5.1 and Propositions 5.4 and 5.6 we obtain:

Theorem 5.8. *For each composition* $\mathcal{I} \stackrel{\mathrm{def}}{=} (n_1, \ldots, n_k)$ *of* n*, define* m_i, d_i *by* (5.3) *and let* $\mathcal{W}(\mathcal{I})$ *denote the set*

$$\left\{ (A_1, \ldots, A_n) : \forall i, \text{ (5.4) } \textit{holds, and } \sum_{i=1}^{k} \left(A_{m_{i-1}+1}^2 - \frac{A_{m_{i-1}+1}^{2n_i+2}}{d_i^2 \gamma_{n_i+1}^{n_i+1}} \right) \le \frac{n}{4} \right\}.$$

If

$$\mathrm{KZS} \subset \bigcup_{\mathcal{I}} \mathcal{W}(\mathcal{I}) \tag{5.7}$$

then Minkowski's conjecture holds in dimension n*.*

Rajinder Hans-Gill has informed the authors that using arguments as in [6, 7], it is possible to verify (5.7) in dimensions up to 7, thus reproving Minkowski's conjecture in these dimensions.

5.2. **Local maxima of covrad.** The aim of this subsection is to prove Corollary 1.3, which shows that in order to establish that all stable lattices in $\mathbb{R}^n$ satisfy the covering radius bound (1.5), it suffices to check this on a finite list of lattices in each dimension $d \le n$.

The function covrad : $\mathcal{L}_n \to \mathbb{R}$ is proper, but nevertheless has local maxima, in the usual sense; that is, lattices $x \in \mathcal{L}_n$ for which there is a neighborhood $\mathcal{U}$ of x in $\mathcal{L}_n$ such that for all $x' \in \mathcal{U}$ we have $\mathrm{covrad}(x') \le \mathrm{covrad}(x)$. Dutour-Sikirić, Schürmann and Vallentin [3] gave a geometric characterization of lattices which are local maxima of the function covrad, and showed that there are finitely many in each dimension. Corollary 1.3 asserts that Minkowski's conjecture would follow if all local maxima of covrad satisfy the bound (1.5).

Proof of Corollary 1.3. We prove by induction on n that any stable lattice satisfies the bound (1.5) and apply Corollary 5.1. Let $\mathcal{S}$ denote the set of stable lattices in $\mathcal{L}_n$. It is compact so the function covrad attains a maximum on $\mathcal{S}$, and it suffices to show that this maximum is at most $\frac{\sqrt{n}}{2}$. Let $x \in \mathcal{S}$ be a point at which the maximum is attained. If x is an interior point of $\mathcal{S}$ then necessarily x is a local maximum for covrad and the required bound holds by hypothesis. Otherwise, there is a sequence $x_j \to x$ such that $x_j \in \mathcal{L}_n \smallsetminus \mathcal{S}$; thus each x_j contains a discrete subgroup Λ_j with $|\Lambda_j| < 1$ and $r(\Lambda_j) < n$. Passing to a subsequence we may assume that that $r(\Lambda_j) = k < n$ is the same for all j, and Λ_j converges to a discrete subgroup Λ of x. Since x is stable we must have $|\Lambda| = 1$. Let $\pi : \mathbb{R}^n \to (\mathrm{span}\Lambda)^{\perp}$ by the orthogonal projection and let $\Lambda' \stackrel{\mathrm{def}}{=} \pi(x)$.

It suffices to show that both Λ and Λ' are stable. Indeed, if this holds then by the induction hypothesis, both Λ and Λ' satisfy (1.5) in their respective dimensions $k, n-k$, and by Lemma 5.3, so does x. To see that Λ is stable, note that any subgroup $\Lambda_0 \subset \Lambda$ is also a subgroup of x, and since x is stable, it satisfies $|\Lambda_0| \geq 1$. To see that Λ' is stable, note that if $\Lambda_0 \subset \Lambda'$ then $\widetilde{\Lambda_0} \stackrel{\mathrm{def}}{=} x \cap \pi^{-1}(\Lambda_0)$ is a discrete subgroup of x so satisfies $|\widetilde{\Lambda_0}| \geq 1$. Since $|\Lambda| = 1$ and π is orthogonal, we argue as in the proof of (2.3) to obtain

$$1 \leq |\widetilde{\Lambda_0}| = |\Lambda| \cdot |\Lambda_0| = |\Lambda_0|,$$

so Λ' is also stable, as required. □

In [3], it was shown that there is a unique local maximum for covrad in dimension 1, none in dimensions 2–5, and a unique one in dimension 6. Local maxima of covrad in dimension 7 are classified in the manuscript [2]; there are 2 such lattices. Thus in total, in dimensions $n \leq 7$ there are 4 local maxima of the function covrad. We were informed by Mathieu Dutour-Sikirić that these lattices all satisfy the covering radius bound (1.5). Thus Corollary 1.3 yields another proof of Minkowski's conjecture, in dimensions $n \leq 7$. In [4] and infinite list of lattices, one in each dimension $n \geq 6$, is defined. It was shown in [3, §7], that each of these lattices (denoted there by $[L_n, Q_n]$) is a local maximum for the function covrad, and satisfies the bound (1.5). Dutour-Sikirić has conjectured:

Conjecture 5.9 (M. Dutour-Sikirić)**.** For each $n \geq 6$, the lattice $[L_n, Q_n]$ has the largest covering radius among all local maxima in dimension n.

In light of Corollary 1.3, the validity of Conjecture 5.9 would imply Minkowski's conjecture in all dimensions.

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