ON THE MORDELL-GRUBER SPECTRUM

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Abstract. We investigate the Mordell constant of certain families of lattices, in particular, of lattices arising from totally real fields. We define the almost sure value κ_μ of the Mordell constant with respect to certain homogeneous measures on the space of lattices, and establish a strict inequality $\kappa_{\mu_1} < \kappa_{\mu_2}$ when the μ_i are finite and $\mathrm{supp}(\mu_1) \subsetneq \mathrm{supp}(\mu_2)$. In combination with known results regarding the dynamics of the diagonal group we obtain isolation results as well as information regarding accumulation points of the Mordell-Gruber spectrum, extending previous work of Gruber and Ramharter. One of the main tools we develop is the associated algebra, an algebraic invariant attached to the orbit of a lattice under a block group, which can be used to characterize closed and finite volume orbits.

1. Introduction

1.1. The Mordell constant of a lattice. Let $\Lambda \subset \mathbb{R}^n$ be a lattice. By a *symmetric box* in $\mathbb{R}^n$ we mean a set of the form $[-a_1, a_1] \times \cdots \times [-a_n, a_n]$, and we say that a symmetric box is *admissible for* Λ if it contains no nonzero points of Λ in its interior. The *Mordell constant* of Λ is defined to be

$$\kappa(\Lambda) \overset{\mathrm{def}}{=} \sup_{\mathcal{B}} \frac{\mathrm{Vol}(\mathcal{B})}{2^n \mathrm{Vol}(\Lambda)}, \tag{1}$$

where the supremum is taken over symmetric boxes $\mathcal{B}$ which are admissible for Λ, and where $\mathrm{Vol}(\mathcal{B})$ denotes the volume of $\mathcal{B}$ and $\mathrm{Vol}(\Lambda)$ denotes the volume of a fundamental domain for Λ. The purpose of this paper is to study the quantity $\kappa(\cdot)$, as a function on the space of lattices; in particular, to study its image, which we call the *Mordell-Gruber spectrum*, its generic values, and isolation properties. Research on these questions stems from the so-called 'Mordell inverse problem' [M] and their in-depth study was carried out in a number of papers, notably those of Gruber and Ramharter. We refer to [GL, Chap. 3] for a detailed history, and give more precise references to the literature below.

Since the function $\kappa(\Lambda)$ is invariant under homotheties, there is no loss of generality in restricting our attention to unimodular lattices (i.e. lattices with $\mathrm{Vol}(\Lambda) = 1$). We will denote the space of unimodular lattices of dimension n by $\mathcal{L}_n$. It is equipped with the transitive action of the group $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_n(\mathbb{R})$, and the function κ is invariant under the action of the subgroup A of diagonal matrices in G with positive diagonal entries[1], since this action permutes symmetric boxes. We will prove new results as well as apply known ones about this A-action on $\mathcal{L}_n$ to derive consequences for κ — see [EL] for a survey of the recent progress in the study of this action.

1.2. **Homogeneous measures and intermediate lattices.** From the dynamical point of view it is natural to study homogeneous A-invariant measures on $\mathcal{L}_n$ which we now define. Let $H\Lambda \subset \mathcal{L}_n$ be closed orbit of a real algebraic subgroup $H \subset G$. We will see in Proposition 2.6 that the orbit $H\Lambda$ supports a locally finite H-invariant measure which is unique up to scaling.

Definition 1.1. Given an A-invariant closed orbit $H\Lambda \subset \mathcal{L}_n$ of a closed connected real algebraic subgroup $H \subset G$ we refer to the H-invariant locally finite measure supported on $H\Lambda$ as the *homogeneous measure* associated with the orbit $H\Lambda$ and denote it by $\mu_{H\Lambda}$. The closed orbit $H\Lambda$ will be referred to as the *homogeneous space* corresponding to the measure.

We emphasize that we allow our homogeneous spaces to be of infinite measure; when the measure $\mu_{H\Lambda}$ is finite we say that the orbit $H\Lambda$ is *of finite volume*. It is well known that the orbit $G\Lambda = \mathcal{L}_n$ is of finite volume and we denote the corresponding (unique) G-invariant probability measure by $\mu_{\mathcal{L}_n}$. The starting point of our discussion is the following

Theorem 1.2. *Let μ be a homogeneous A-invariant measure on $\mathcal{L}_n$. Then for μ-almost any Λ*

$$\kappa(\Lambda) = \max\{\kappa(\Lambda') : \Lambda' \textit{ is in the support of } \mu\}. \tag{2}$$

Theorem 1.2 is a standard consequence of the ergodicity of the A-action and is proved in §3. We note that a well-known conjecture of Margulis [M2] asserts that in dimension $n \geq 3$ any A-invariant and A-ergodic probability measure on $\mathcal{L}_n$ is homogeneous.

[1]For notational convenience we will not work with the full group of linear maps preserving κ, which besides A, also contains non-positive diagonal matrices and permutations of the coordinates.

The following consequence of Theorem 1.2 answers a question of Gruber, and improves on previous results of Gruber and Ramharter [GR, R1, R2]. Note that Minkowski's convex body Theorem implies that $\kappa(\Lambda) \leq 1$ for any Λ, this upper bound being attained by $\Lambda = \mathbb{Z}^n$. Therefore taking $\mu = \mu_{\mathcal{L}_n}$ we obtain:

Corollary 1.3. *With respect to $\mu_{\mathcal{L}_n}$, almost every lattice has Mordell constant equal to 1.*

A natural question is the existence and characterization of lattices with Mordell constant strictly smaller than 1. In view of Theorem 1.2, given a homogeneous A-invariant measure μ, it makes sense to define *the generic value* κ_μ to be the almost sure value of κ with respect to μ. One of the main results of this paper is the following:

Theorem 1.4. *Let μ_1, μ_2 be two A-invariant homogeneous measures such that μ_1 is finite and* $\mathrm{supp}(\mu_1) \subsetneq \mathrm{supp}(\mu_2)$. *Then $\kappa_{\mu_1} < \kappa_{\mu_2}$.*

In §6 we show by examples that the hypothesis that μ_1 is finite in Theorem 1.4 is essential. Nevertheless, we will establish Theorem 6.1 which extends Theorem 1.4 to the case of A-invariant homogeneous measures which are not necessarily finite, under a suitable additional assumption.

In order to prove Theorems 1.4 and 6.1 we will study homogeneous A-invariant measures. As will be shown in Proposition 3.2, the groups H that give rise to homogeneous A-invariant measures are *block groups* obtained by choosing a partition $\mathcal{P} = \sqcup_1^r Q_\ell$ of $\{1 \dots n\}$ and defining

$$H(\mathcal{P}) = \{(g_{ij}) \in G : g_{ij} \neq 0 \Rightarrow i, j \in Q_\ell \text{ for some } \ell\}^\circ \tag{3}$$

(where L° is the connected component of the identity in the group L). In §4 we study orbits of block groups in detail. We attach to each orbit $H\Lambda$ of a block group an algebraic invariant we refer to as the *associated algebra* which is a finite dimensional $\mathbb{Q}$-algebra. Simple algebraic properties of the associated algebra allow us to determine whether the orbit is closed or of finite volume (see Theorem 4.2). Whenever we have a containment $H_1\Lambda \subset H_2\Lambda$ of orbits as above, we have a reverse inclusion of the associated algebras and the condition which allows us to generalize Theorem 1.4 is a simple algebraic property of the inclusion of the associated algebras.

Definition 1.5. A lattice $\Lambda \in \mathcal{L}_n$ is said to be *intermediate* (resp. *intermediate of finite volume type*) if it belongs to an A-invariant homogeneous space (resp. of finite volume) $H\Lambda$ which is strictly contained in $\mathcal{L}_n$.

The case $H = A$ will be of particular interest.

Definition 1.6. A lattice Λ for which $A\Lambda$ is closed will be referred to as an *algebra lattice*. If furthermore the orbit is of finite volume (equivalently, if it is compact) then the lattice is said to be a *number field lattice*.

Corollary 5.3 will justify our choice of terminology. It shows that a lattice Λ is an algebra (resp. number field) lattice if and only if the associated algebra is n-dimensional (resp. is an n-dimensional field) over the rationals.

Combining Theorems 1.2 and 1.4 we obtain the following

Corollary 1.7. *Let $\Lambda \in \mathcal{L}_n$ be an intermediate lattice of finite volume type, then $\kappa(\Lambda) < 1$.*

Corollary 1.7 is probably not new for number field lattices (see e.g. [R1]) but we could not locate a suitable reference.

1.3. **Isolation results.** The results below concern isolation properties that follow from Theorem 1.4 and a rigidity result for the A-action in dimension $n \geq 3$ (Theorem 7.1).

Definition 1.8. Let Λ be a lattice and let $\varepsilon_0 > 0$. We say that Λ is ε_0*-isolated* if for any $0 < \varepsilon < \varepsilon_0$ there is a neighborhood $\mathcal{U}$ of Λ in $\mathcal{L}_n$, such that for any $\Lambda' \in \mathcal{U} \smallsetminus A\Lambda$, $\kappa(\Lambda') > \kappa(\Lambda) + \varepsilon$.

We say that Λ is *locally isolated* if it is ε_0-isolated for some $\varepsilon_0 > 0$, and that Λ is *strongly isolated* if it is ε_0-isolated for $\varepsilon_0 = 1 - \kappa(\Lambda)$.

Theorem 1.9. *Let $n \geq 3$, and let Λ be a number field lattice, associated with the degree n number field F. Then Λ is locally isolated. Moreover Λ is strongly isolated if and only if there are no intermediate fields $\mathbb{Q} \subsetneq K \subsetneq F$.*

Theorem 1.9 extends results of Ramharter [R1], who shows local isolation under an additional assumption, but does not require $n \geq 3$. Our methods crucially rely on the hypothesis $n \geq 3$. An immediate consequence is:

Corollary 1.10. *If $n \geq 3$ is prime, then any number field lattice in $\mathbb{R}^n$ is strongly isolated.*

We remark that when n is prime, an intermediate lattice of finite volume type is automatically a number field lattice. Extending another result of [R1] we show:

Corollary 1.11. *For any $n \geq 3$, the set of strongly isolated lattices is dense in $\mathcal{L}_n$.*

The fundamental difference between the cases $n = 2$ and $n \geq 3$ is highlighted in the following:

Theorem 1.12. *In dimension $n = 2$ there are no strongly isolated lattices.*

Theorem 1.12 relies on work of Gruber [G]. In contrast with Theorem 1.9, intermediate lattices which are not number field lattices are typically not isolated. In §7 we will define a notion of 'local relative isolation' and prove:

Theorem 1.13. *Let μ be an A-invariant homgeneous probability measure corresponding to a finite volume orbit $H\Lambda$ with $A \subsetneq H \subsetneq G$. Then almost any lattice with respect to μ is not locally isolated but is locally isolated relative to H.*

1.4. **The reduced Mordell-Gruber spectrum.** We denote by $\mathbf{MG}_n$ the *Mordell-Gruber spectrum*, which is the set of numbers $\kappa(\Lambda)$ where Λ ranges over all lattices in dimension n. We briefly summarize some of the known facts about the Mordell-Gruber spectrum. Siegel (see [GL]) showed that $\kappa_n \stackrel{\mathrm{def}}{=} \inf \mathbf{MG}_n > 0$. Many things are known about $\mathbf{MG}_2$, see [G]. The values κ_2 and κ_3 are known, the latter by a difficult work of Ramharter [R3]. In [R3] Ramharter also showed that κ_3 belongs to $\mathbf{MG}_3$ and is an isolated[2] point. Various lower bounds on κ_n have been proved by various authors, and recently [SW] the authors obtained the lower bound $\kappa_n \geq n^{-n/2}$.

We wish to study accumulation points of $\mathbf{MG}_n$. There is a simple trick to generate such points which we now describe. As we explain in §8, it can be deduced from results of Gruber that $\mathbf{MG}_2$ has many accumulation points. We say that a lattice $\Lambda \subset \mathbb{R}^n$ is *decomposable* if $n = n_1+n_2$, $n_i > 0$, and $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ is the direct sum decomposition corresponding to partitioning the coordinates into subsets of sizes n_1 and n_2, and we can write $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Lambda_i = \Lambda \cap \mathbb{R}^{n_i}$. In this case we clearly have $\kappa(\Lambda) = \kappa(\Lambda_1)\kappa(\Lambda_2)$. Taking direct sums with $\mathbb{Z}^{n_2}$ we get embeddings $\mathbf{MG}_{n_1} \hookrightarrow \mathbf{MG}_n$ for any $n_1 < n$. We are interested in the part of the spectrum not arising in this way. That is, we define the *reduced Mordell-Gruber spectrum* to be

$$\widehat{\mathbf{MG}}_n \stackrel{\mathrm{def}}{=} \{\kappa(\Lambda) : \Lambda \subset \mathbb{R}^n \text{ a lattice which is not decomposable}\}.$$

As will be seen in Proposition 5.10, number field lattices are never decomposable. We are interested in the existence of accumulation points of $\widehat{\mathbf{MG}}_n$.

[2]The isolation of a number in a subset of $\mathbb{R}$ should not be confused with the isolation property of Definition 1.8.

Theorem 1.14. *Let μ be an A-invariant homogeneous probability measure corresponding to a finite volume orbit $H\Lambda$ with $A \subsetneq H$. Then there is a sequence of number field lattices Λ_k for which $\kappa(\Lambda_k) \nearrow \kappa_\mu$. In particular, κ_μ is not an isolated point of $\widehat{\mathbf{MG}}_n$.*

Taking μ to be the Haar measure we obtain

Corollary 1.15. *For any n there is a sequence (Λ_k) of number field lattices for which $\kappa(\Lambda_k) \nearrow 1$. In particular 1 is not an isolated point of $\widehat{\mathbf{MG}}_n$.*

Given a subset $M \subset \mathbb{R}$, we denote $M^{(0)} \stackrel{\mathrm{def}}{=} M$, and by $M^{(k+1)}$ the elements of $M^{(k)}$ which are limits of strictly increasing sequences in $M^{(k)}$.

Theorem 1.16. *For any natural number t, there is n so that $1 \in \widehat{\mathbf{MG}}_n^{(t)}$.*

1.5. **Organization of the paper.** In sections §2 and §3 we recall some standard results and prove some useful results about closed orbits for actions of algebraic groups on $\mathcal{L}_n$. From these we deduce Theorem 1.2. In §4 we introduce the *associated algebra* of a lattice and characterize intermediate lattices in terms of its algebraic properties. As we explain in §5, the associated algebra of a lattice Λ can be used to classify all A-invariant homogeneous subsets containing Λ. Moreover in §5.2 we show how to explicitly construct all intermediate lattices. The proof of Theorem 1.4 is given in §6 and of the isolation results in §7. In §8 we recall results of Gruber and Ramharter for dimension $n = 2$, give some more information about $\mathbf{MG}_2$, and prove Theorem 1.12.

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2. Generalities

The following two propositions are standard and explained in [GL].

Proposition 2.1. *The function $\kappa : \mathcal{L}_n \to \mathbb{R}$ has the following properties:*

(i) *For all Λ, $\kappa(\Lambda) \le 1$.*
(ii) *If $\Lambda_k \to \Lambda$ then $\kappa(\Lambda) \le \liminf \kappa(\Lambda_k)$; i.e. κ is lower semi-continuous.*
(iii) *For all Λ and all $a \in A$, $\kappa(a\Lambda) = \kappa(\Lambda)$.*
(iv) *$\kappa(\mathbb{Z}^n) = 1$.*

Proposition 2.2 (Mahler's compactness criterion)**.** *A subset $X \subset \mathcal{L}_n$ is bounded (i.e. has compact closure) if and only if there is a neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^n$ such that for any $\Lambda \in X$, $\Lambda \cap \mathcal{U} = \{0\}$.*

Corollary 2.3. *If $\Lambda \in \mathcal{L}_n$ and $\mathcal{B}_0$ is a symmetric cube whose volume is smaller than $2^n\kappa(\Lambda)$, then there is $a \in A$ such that $\mathcal{B}_0$ is admissible for $a\Lambda$. In particular for any $\kappa_0 > 0$ there is a compact $K \subset \mathcal{L}_n$ such that for any $\Lambda \in \mathcal{L}_n$ with $\kappa(\Lambda) \ge \kappa_0$, there is $a \in A$ such that $a\Lambda \in K$.*

Proof. For the first assertion, let $\mathcal{B}$ be an admissible symmetric box such that $\mathrm{Vol}(\mathcal{B}) > \mathrm{Vol}(\mathcal{B}_0)$, and let $a \in A$ such that $a\mathcal{B}$ is a cube symmetric about the origin. By considering volumes we see that $\mathcal{B}_0 \subset a\mathcal{B}$. This proves the first assertion. The second assertion follows via Proposition 2.2. $\square$

2.1. **Algebraic groups and $\mathbb{Q}$-structures.** We use the term *real algebraic group* to refer to a finite index subgroup of the set of real points of a Zariski closed group. Often we simply say *algebraic group*. With this terminology an algebraic group need not be Zariski closed but is of finite index in its Zariski closure. In the remainder of this section we will recall several classical results about algebraic groups and lattices in Lie groups. We refer the reader to [Rag] for more details and pointers to the literature. In this paper, we have preferred a concrete point of view so we will work throughout with subgroups of $G = \mathrm{SL}_n(\mathbb{R})$ and with the space $\mathcal{L}_n \cong \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, rather than the more general setup where G is a real algebraic group and G/Γ is the quotient of G by a lattice Γ. All the results we state below are valid in this more general context.

Given a lattice $\Lambda \in \mathcal{L}_n$ we denote $V_\Lambda \stackrel{\mathrm{def}}{=} \mathrm{span}_{\mathbb{Q}}\Lambda$. Note that V_Λ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}^n$ such that $V_\Lambda \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^n$, but V_Λ need not coincide with the standard $\mathbb{Q}$-structure $\mathbb{Q}^n$. We say that a matrix $g \in G$ is Λ-rational if $gV_\Lambda \subset V_\Lambda$; the reader may verify that $g \in G(\mathbb{Q})$ if and only if g is $\mathbb{Z}^n$-rational. As in [Rag, Preliminaries, §2] one uses a $\mathbb{Q}$-structure on $\mathbb{R}^n$ to define $\mathbb{Q}$-algebraic subgroups of $\mathrm{SL}_n(\mathbb{R})$. If we use the $\mathbb{Q}$-structure of V_Λ we will say that such a subgroup H is *defined* *over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-structure induced by Λ.* We will use two characterizations of such subgroups. They are the algebraic groups H whose Λ-rational points are Zariski dense in their Zariski closure; they are also the algebraic groups H such that for any $g \in G$ for which $g\mathbb{Z}^n = \Lambda$, the conjugate $g^{-1}Hg$ is a $\mathbb{Q}$-subgroup of G (with respect to the standard $\mathbb{Q}$-structure $\mathbb{Q}^n$). See [Rag] for definitions of morphisms defined over $\mathbb{Q}$ and $\mathbb{Q}$-characters.

We recall the following classical fact, see e.g. [Rag, Chap. XIII].

Proposition 2.4 (Borel Harish-Chandra)**.** *Let $H \subset G$ be an algebraic group defined over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-structure induced by $\Lambda \in \mathcal{L}_n$. Then the orbit $H\Lambda \subset \mathcal{L}_n$ is of finite volume if and only if H° has no non-trivial $\mathbb{Q}$-characters. In particular, if H is semisimple or is generated by unipotent elements then $H\Lambda$ is of finite volume.*

2.2. **Closed orbits of real algebraic subgroups.** The following observation is useful.

Proposition 2.5. *Let $\Lambda \in \mathcal{L}_n$ and let H be an algebraic subgroup of G. Denote by H_Λ the stabilizer of Λ in H and by H_0 the Zariski closure of H_Λ. Then H_Λ is a lattice in H_0. Moreover, if the orbit $H\Lambda$ is closed in $\mathcal{L}_n$ then the connected component H_0° of the identity in H_0 contains the unipotent elements of H.*

Proof. We use the $\mathbb{Q}$-structure induced by Λ. For any $\mathbb{Q}$-character χ of an algebraic group containing H_Λ, the image $\chi(H_\Lambda)$ is bounded below by a bounded denominators argument. Therefore $\chi(H_\Lambda)$ is finite, which implies the finiteness of $\chi(H_0)$. In particular, χ is trivial on $(H_0)^\circ$. By Proposition 2.4, $(H_0)_\Lambda$ is a lattice in H_0.

Since H, H_0 are commensurable, the same holds for $H_\Lambda, (H_0)_\Lambda$. It follows that H_Λ is of finite index in $(H_0)_\Lambda$ and thus H_Λ is a lattice in H_0 as well.

Now suppose u is a unipotent element of H and suppose $H\Lambda$ is closed. There is a one-parameter unipotent subgroup $\{u(t)\} \subset H$ such that $u = u(1)$. By a classical result of Margulis [M1], the trajectory $\{u(t)\Lambda : t \geq 0\}$ is not divergent. The orbit map $hH_\Lambda \mapsto h\Lambda$ is proper since we have assumed that $H\Lambda$ is closed, and this implies that the orbit $\{u(t)H_\Lambda : t \geq 0\}$ is non-divergent in the quotient H/H_Λ. We have an H-equivariant factor map $H/H_\Lambda \to H/H_0$, so the orbit $\{u(t)H_0 : t \geq 0\}$ is non-divergent in H/H_0, which is an algebraic variety on which a unipotent trajectory is either a fixed point or is divergent. This implies that $\{u(t)\} \subset H_0$, and by connectedness, $\{u(t)\} \subset H_0^\circ$. $\square$

As we allow homogeneous subspaces of infinite measure, we need the following fact:

Proposition 2.6. *Let $H \subset G$ be a real algebraic group and $\Lambda \in \mathcal{L}_n$ such that $H\Lambda$ is closed. Then there is an H-invariant locally finite measure on $H\Lambda$. This measure is unique up to scaling.*

We remark that in contrast to finite volume homogeneous spaces, for infinite volume homogeneous spaces H need not be unimodular. We also remark that in the statement of the Proposition one may replace $\mathcal{L}_n$ with any homogeneous space G/Γ where G is a real algebraic group and Γ is a lattice.

Proof. Let H_Λ denote the stabilizer of Λ in H. There is an injective *orbit map*

$$H/H_\Lambda \to \mathcal{L}_n, \quad hH_\Lambda \mapsto h\Lambda,$$

and the assumption that $H\Lambda$ is closed implies that this map is a homeomorphism onto its image. So it is enough to prove that there is an H-invariant locally finite measure on H/H_Λ. For a Lie group L, let Δ_L denote its modular function. In light of general facts about quotients of Lie groups (see e.g. [Rag, Chapter 1]), it is enough to show that

$$\Delta_H|_{H_\Lambda} = \Delta_{H_\Lambda}. \tag{4}$$

Since H_Λ is discrete, it is unimodular, so Δ_{H_Λ} is trivial. So we need to show that the restriction of Δ_H to H_Λ is trivial. We have the explicit formula

$$\Delta_H(h) = |\det \mathrm{Ad}(h)|_{\mathfrak{u}}|,$$

where $\mathfrak{u}$ is the Lie algebra of the unipotent radical U of H. In other words $\Delta_H(h)$ is the multiplicative factor by which conjugation by h multiplies the Haar measure on U.

We will now show that $\Gamma \stackrel{\mathrm{def}}{=} U \cap H_\Lambda$ is a lattice in U. Indeed, let H_0 be as in Proposition 2.5, let H' be the subgroup of H generated by the unipotent elements of H, and let U_0, U' denote respectively the unipotent radicals of H_0, H'. By Proposition 2.5,

$$H' \subset (H_0)^\circ \subset H,$$

which implies

$$U \subset U_0 \subset U'.$$

On the other hand, H' is a characteristic subgroup of H, and hence so is its unipotent radical. Therefore U' is normal in H, which implies that $U = U'$. In particular $U = U_0$. It now follows from [Rag, Cor. 8.28] that $\Gamma \cap U$ is a lattice in U, as required. Since conjugation by elements of H_Λ preserves both U and H_Λ, it fixes Γ and so fixes the covolume of U/Γ. This implies (4). ◻

3. Ergodicity and consequences

In this section we prove Theorem 1.2 by establishing the ergodicity of the A-action with respect to homogeneous A-invariant measures (see Proposition 3.6). In order to establish this, we study in some detail the structure of homogeneous A-invariant spaces in $\mathcal{L}_n$.

Let X be a locally compact topological space, let μ be a locally finite Borel measure on X, and let A be a group acting continuously on X preserving μ. The action is called *ergodic* if any invariant set is either of zero measure, or its complement is of zero measure. Let $\mathrm{supp}(\mu)$ denote the topological support of μ. Then it is well-known (see e.g. [Z]) that when the action is ergodic, any A-invariant measurable function $X \to \mathbb{R}$ is almost everywhere constant and for almost every $x \in X$, the orbit Ax is dense in $\mathrm{supp}(\mu)$.

Theorem 3.1. *Let $X = \overline{A\Lambda_0}$ be an orbit-closure for the A-action on $\mathcal{L}_n$. Then*

$$\kappa(\Lambda_0) = \sup\{\kappa(\Lambda) : \Lambda \in X\}.$$

In particular, if μ is an A-invariant and A-ergodic measure, then for μ-almost every Λ_0 we have

$$\kappa_\mu \stackrel{\mathrm{def}}{=} \sup\{\kappa(\Lambda) : \Lambda \in \mathrm{supp}(\mu)\} = \kappa(\Lambda_0) \tag{5}$$

and so the supremum in (5) is attained. Moreover $\mathrm{supp}(\mu)$ contains a lattice $\Lambda_{\max}$ with $\kappa(\Lambda_{\max}) = \kappa_\mu$ such that the cube $\mathcal{C}$ of volume $2^n\kappa_\mu$ is admissible for $\Lambda_{\max}$ (so the supremum in (1) is attained for $\Lambda = \Lambda_{\max}$, $\mathcal{B} = \mathcal{C}$).

Proof. By Proposition 2.1(ii),(iii), for any $\Lambda \in X$, $\kappa(\Lambda) \le \kappa(\Lambda_0)$. This proves the first assertion. The second one follows taking $X = \mathrm{supp}(\mu)$ and recalling that almost every A-orbit in X is dense.

For the last assertion, let $\Lambda \in \mathrm{supp}(\mu)$ with $\kappa(\Lambda) = \kappa_\mu$ and let C_k be a sequence of symmetric cubes with $\mathrm{Vol}(C_k) \nearrow 2^n\kappa(\Lambda)$. For each k, by Corollary 2.3 there is $a_k \in A$ so that C_k is admissible for $a_k(\Lambda)$ and the sequence $\{a_k\Lambda\}$ is contained in a bounded subset of $\mathcal{L}_n$. Let $\Lambda_{\max}$ be a limit of a converging subsequence of $\{a_k\Lambda\}$. Then $\Lambda_{\max} \in \mathrm{supp}(\mu)$ since $\mathrm{supp}(\mu)$ is closed and A-invariant. Moreover by construction, the cube of volume $2^n\kappa_\mu$ is admissible for $\Lambda_{\max}$. This implies that $\Lambda_{\max}$ has the required properties. $\square$

3.1. Block groups.

Given a partition of the indices $\{1, \dots, n\}$

$$\mathcal{P} \stackrel{\mathrm{def}}{=} \left(\{1, \dots, n\} = \bigsqcup Q_\ell\right), \tag{6}$$

we define the *block group* corresponding to $\mathcal{P}$ to be the connected subgroup $H = H(\mathcal{P})$ of G whose Lie algebra is

$$\mathfrak{h} = \mathfrak{a} \oplus \bigoplus_{\ell} \bigoplus_{s,t \in Q_\ell} \mathfrak{g}_{st} \tag{7}$$

where $\mathfrak{a}$ is the Lie algebra of A and $\mathfrak{g}_{st}$ is the one-dimensional Lie algebra spanned by the matrix with 1 in the entry (s,t) and 0 elsewhere (note that we always have $A \subset H$). We refer to the elements Q_ℓ of $\mathcal{P}$ as the *blocks* of the partition and denote by $|\mathcal{P}|$ the number of blocks. When the blocks are of equal size we say that $\mathcal{P}$ is an *equiblock partition* and $H(\mathcal{P})$ is an *equiblock group*. Given a partition $\mathcal{P}$ we shall denote by $\sim_{\mathcal{P}}$ the equivalence relation it defines on $\{1, \dots, n\}$. For example, up to permutations of indices, the three equiblock partitions for $n = 4$ are

$$\mathcal{P}_0 = (\{1\}, \{2\}, \{3\}, \{4\}), \quad \mathcal{P}_2 = (\{1,2\}, \{3,4\}), \ \mathcal{P}_2 = (\{1,2,3,4\}),$$

and the corresponding equiblock groups are

$$H(\mathcal{P}_0) = A, \ H(\mathcal{P}_1) = \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix} \cap G, \ H(\mathcal{P}_2) = G.$$

Our interest in block groups is explained by the following:

Proposition 3.2. *Let $H\Lambda \subset \mathcal{L}_n$ be a homogeneous space (i.e. a closed orbit of a closed connected subgroup $H \subset G$). Then if $H\Lambda$ is A-invariant, then $A \subset H$ and $H = H(\mathcal{P})$ for some partition $\mathcal{P}$.*

We will use the following simple Lemma whose proof is left to the reader.

Lemma 3.3. *If $H_1\Lambda \subset H_2\Lambda$ is a containment of two orbits in $\mathcal{L}_n$ of closed groups H_1, H_2 and H_1 is connected, then $H_1 \subset H_2$.*

□

Proof of Proposition 3.2. The fact that $A \subset H$ follows from Lemma 3.3. Note that if $H \subset G$ is a closed connected subgroup containing the diagonal group then in the above notation, the Lie algebra of H satisfies $\mathfrak{h} = \mathfrak{a} \oplus \bigoplus \mathfrak{g}_{st}$, where the sum is taken over some subset of the set of pairs (s,t). Since $\mathfrak{h}$ is a Lie algebra, $\mathfrak{g}_{s,t}, \mathfrak{g}_{t,u} \subset \mathfrak{h}$ implies $\mathfrak{g}_{s,u} \subset \mathfrak{h}$. We need to show that

$$\mathfrak{g}_{s,t} \subset \mathfrak{h} \implies \mathfrak{g}_{t,s} \subset \mathfrak{h}.$$

Let $u_{st} \subset H$ be the one parameter unipotent group with Lie algebra $\mathfrak{g}_{st}$. There exists a one-parameter subgroup $A_0 \subset A$, such that the group B generated by A_0, $\{u_{ts}(x)\}$ is the Borel subgroup of the copy of $\mathrm{SL}_2(\mathbb{R}) \subset G$ which is generated by the two groups $u_{ts}(x), u_{st}(x)$. We denote this copy of $\mathrm{SL}_2(\mathbb{R})$ by H_0. As $H\Lambda$ is assumed to be closed we have $H\Lambda \supset \overline{B\Lambda}$. By the work of Ratner (see [Ra] for a short proof) we have $\overline{H_0\Lambda} = \overline{B\Lambda}$ and so we conclude that $\{u_{st}(x)\Lambda : x \in \mathbb{R}\} \subset H\Lambda$. By Lemma 3.3, $\{u_{st}(x)\} \subset H$ as desired. □

We will see in Corollary 5.4 that in the case $H\Lambda$ is of finite volume, there are further restrictions on the partition $\mathcal{P}$ in the above proposition.

3.2. **Structure of A-invariant homogeneous measures.** Let $H = H(\mathcal{P})$ be a block group. Then it can be written as a direct product $H(\mathcal{P}) = Z(\mathcal{P}) \cdot S(\mathcal{P})$, where

$$Z(\mathcal{P}) \stackrel{\mathrm{def}}{=} \{a \in A : a \text{ centralizes } H(\mathcal{P})\}$$

(that is, $Z(\mathcal{P})$ consists of the positive diagonal matrices in $H(\mathcal{P})$ that have constant eigenvalues along the blocks of $\mathcal{P}$), and $S(\mathcal{P})$ is the commutator group of $H(\mathcal{P})$. More concretely, $S(\mathcal{P})$ is the semisimple group of matrices having the block structure given by $\mathcal{P}$ with the further requirement that the determinant of each block is 1.

The following proposition shows that an A-invariant homogeneous measure has a simple product structure.

Proposition 3.4. *Let $H = H(\mathcal{P})$ be a block group, $H\Lambda$ an A-invariant homogeneous space, and $\mu = \mu_{H\Lambda}$ the corresponding A-invariant measure. Then there is a decomposition of $Z = Z(\mathcal{P})$ as a direct product $Z = Z_s \cdot Z_a$ such that*

(1) *If $H_1 = Z_a \cdot S$, where $S = S(\mathcal{P})$, then $H_1\Lambda$ is of finite volume.*
(2) *The map*
$$Z_s \to Z_s\Lambda, \quad z \mapsto z\Lambda$$
is proper, and a homeomorphism onto its image. In particular, the orbit $Z_s\Lambda$ is divergent.
(3) *The map*
$$Z_s \times H_1\Lambda \to H\Lambda, \quad (z, h\Lambda) \mapsto zh\Lambda$$
is a homeomorphism onto its image, under which μ is identified with $\nu \times \mu_{H_1\Lambda}$, where ν is Haar measure on Z_s.

Remark 3.5. Note that, since Z is central in H, the conclusions (1) and (2) hold for any $\Lambda' \in H\Lambda$.

Proof. Let $H_1 = (H_0)^\circ \subset H$ where H_0 denotes the Zariski closure of H_Λ. It follows from Proposition 2.5 that the orbit $H_1\Lambda$ is of finite volume, and also that $S \subset H_1$. Since

$$S \subset H_1 \subset Z \cdot S = H$$

we find that $H_1 = Z_a \cdot S$, where $Z_a \overset{\text{def}}{=} H_1 \cap Z$ which establishes (1).

Let Z_s be any direct complement of Z_a in Z; that is, a subgroup of Z such that $Z = Z_s \cdot Z_a$ (a direct product). Consider the natural embeddings

$$H_1/(H_1)_\Lambda \hookrightarrow H/H_\Lambda \hookrightarrow H\Lambda \subset \mathcal{L}_n. \tag{8}$$

As the orbits $H\Lambda, H_1\Lambda$ are closed, the embeddings in (8) are proper. We claim that the natural map

$$Z_s \times (H_1/(H_1)_\Lambda) \to H/H_\Lambda, (z, h(H_1)_\Lambda) \mapsto zhH_\Lambda, \tag{9}$$

is a homeomorphism. Once this is established, (2) and the first statement of (3) follow. The statement regarding the measures now follows from the uniqueness of an H-invariant measure [Rag, Chap. 1] on H/H_Λ.

We establish (9). Because $H = Z_s \cdot H_1$ the map is clearly onto. It is 1-1 because assuming $(z_1, h_1(H_1)_\Lambda) \neq (z_2, h_2(H_1)_\Lambda)$ than if $z_1h_1H_\Lambda = z_2h_2H_\Lambda$ then since Z_s is central, $z_2^{-1}z_1h_2^{-1}h_1 \in H_\Lambda$. In particular, this element belongs to H_0 - the Zariski closure of H_Λ. It follows that $z_2^{-1}z_1 \in H_0$ and in turn that the one-parameter subgroup generated by it lies in H_0 as well. As this subgroup is connected, it belongs to H_1 and so $z_2^{-1}z_1 \in H_1$. Since $Z_s \cap H_1 = e$ we conclude that $z_1 = z_2$ and finally that $h_1 = h_2$ because of the injectivity on the left of (8). We are thus left to justify the properness of the map (9). Note that $(H_1)_\Lambda < H_\Lambda$ is of finite index and thus, as far as properness is concerned, it is enough to prove properness of $Z_s \times (H_1/(H_1)_\Lambda) \to H/(H_1)_\Lambda$. Because $H = Z_s \cdot H_1$ is a direct product, this is equivalent to saying that the map $Z_s \to H/(H_1)_\Lambda$ is proper. The latter is of course implied by the stronger statement that the map $Z_s \to H/H_1 \cong Z_s$ is proper (in fact a homeomorphism). $\square$

Proposition 3.6. *Let μ be an A-invariant homogeneous measure on $\mathcal{L}_n$ corresponding to the closed orbit $H\Lambda$. Then the A-action is ergodic with respect to μ.*

Proof. We use the notation of Proposition 3.4. Identifying the orbit $H\Lambda$ with the product $Z_s \times H_1\Lambda$ we see that, since $Z_s \subset A$, the statement reduces to the ergodicity of the action of $A \cap H_1$ with respect to the finite H_1-invariant measure $\mu_{H_1\Lambda}$. The latter statement follows from the Howe-Moore Theorem (see e.g. [Z]). $\square$

Proof of Theorem 1.2. The statement follows from Theorem 3.1 and Proposition 3.6. □

Remark 3.7. We use the notation of Proposition 3.4.

(1) It is clear from Proposition 3.4, that the closed orbit $H\Lambda$ is of finite volume if and only if $Z = Z_a$.
(2) The group Z_a in Proposition 3.4 is the center of H_1 and so is a $\mathbb{Q}$-group itself (with respect to the $\mathbb{Q}$-structure induced by Λ). Moreover, it has no non-trivial $\mathbb{Q}$-characters as these will induce corresponding ones on H_1 because H_1 is a direct product $H_1 = Z_a \cdot S$. By the Borel Harish-Chandra Theorem it follows that $Z_a\Lambda$ is of finite volume (which in this case means compact) or in other words, if we denote $Z_\Lambda = \mathrm{Stab}_Z(\Lambda)$ then Z_Λ is a lattice in Z_a. As $Z_a \subset A \cong \mathbb{R}^{n-1}$ we conclude that in particular, the discrete subgroup Z_Λ is a finitely generated free abelian group with $\mathrm{rank}(Z_\Lambda) = \dim Z_a$.
(3) Combining (1),(2) we conclude that the orbit $H\Lambda$ is of finite volume if and only if $\mathrm{rank}(Z_\Lambda) = \dim Z$.

4. Intermediate lattices

We now introduce intermediate lattices, and the homogeneous subspaces they belong to, in detail. This builds on and expands earlier work of several authors, see [LW, T, McM, ELMV]. Our approach is close to that of [McM], in that we emphasize the structure of algebras of matrices associated with a lattice. We introduce for any lattice an *associated algebra*. In §4.3 we characterize intermediate lattices and the homogeneous spaces they belong to by simple algebraic properties of the associated algebra. In §5.2 we explain some constructions of lattices, and show using the aforementioned characterization, that the constructions give rise to all intermediate lattices. In turn, this gives rise to an explicit construction of all homogeneous A-invariant measures. These results will be an important ingredient in the proof of Theorem 1.4.

4.1. **$\mathbb{Q}$-algebras.** Let $F_j,\ j = 1 \dots r$ be number fields and consider the direct sum $B = \oplus_{j=1}^{r} F_j$. Equipped with coordinate-wise addition and multiplication, B is a finite dimensional $\mathbb{Q}$-algebra. By the Artin-Wedderburn Theorem, any commutative finite dimensional semisimple $\mathbb{Q}$-algebra is of the above form.

By a homomorphism between two such algebras we shall mean a map that respects the algebraic operations and sends the identity of one algebra to the identity element of the other. If B is an algebra as above, then an algebra $B' \subset B$ will be referred to as a subalgebra if the inclusion $B' \hookrightarrow B$ is a homomorphism; in particular, B and B' share the same unit. A subalgebra $B' \subset B$ will be referred to as a subfield if it is a field. We emphasize that if $B = \oplus_1^r F_j$ as above, with $r > 1$, then the F_j's are not subalgebras nor subfields.

The theory of algebras of the above form is almost completely analogous to the theory of number fields with only minor adaptations resulting from the fact that we deal with direct sums of number fields. For example it is clear that if $B = \oplus_1^r F_j$ is an n-dimensional $\mathbb{Q}$-algebra, then it has exactly n distinct homomorphisms into $\mathbb{C}$ and those are obtained by first projecting to the components F_j and then composing with the various embeddings of the fields F_j into $\mathbb{C}$.

4.2. **The associated algebra.** Let D denote the algebra of $n \times n$ diagonal real matrices. For $i = 1, \ldots, n$, let $p_i : D \to \mathbb{R}$ be the algebra homomorphism $\mathrm{diag}(d_1, \ldots, d_n) \mapsto d_i$. Given a partition $\mathcal{P}$ as in (6) we denote by $D(\mathcal{P})$ the subalgebra of D defined by

$$D(\mathcal{P}) \stackrel{\mathrm{def}}{=} \{x \in D : p_i(x) = p_j(x) \text{ whenever } i \sim_{\mathcal{P}} j\}.$$

Definition 4.1. Let $\Lambda \subset \mathbb{R}^n$ be a lattice.

(1) We denote $V_\Lambda \stackrel{\mathrm{def}}{=} \mathrm{span}_{\mathbb{Q}}(\Lambda)$.

(2) For any partition $\mathcal{P}$ we define the *associated algebra of* Λ *with respect to* $\mathcal{P}$ to be

$$\mathcal{A}_\Lambda(\mathcal{P}) \stackrel{\mathrm{def}}{=} \{a \in D(\mathcal{P}) : aV_\Lambda \subset V_\Lambda\}.$$

We denote by $\mathcal{P}_0$ the partition into singletons, denote $\mathcal{A}_\Lambda(\mathcal{P}_0)$ simply by $\mathcal{A}_\Lambda$, and refer to it as the *associated algebra* to Λ.

(3) Given a subalgebra $B \subset \mathcal{A}_\Lambda$ we define the *associated partition* $\mathcal{P}_B$ to be the partition of $\{1, \ldots, n\}$ induced by the equivalence relation

$$i \sim j \iff p_i|_B \sim p_j|_B.$$

A partition of the form $\mathcal{P}_B$ will be referred to as an *algebra partition for* Λ and in case $B \subset \mathcal{A}_\Lambda$ is a subfield, as a *field partition for* Λ.

Examples for the case $n = 4$ will be given in §5.4. The following result demonstrates the usefulness of the associated algebra for the study of the A-action on $\mathcal{L}_n$:

Theorem 4.2. *Let $\Lambda \in \mathcal{L}_n$, $\mathcal{P}$ a partition, and $H = H(\mathcal{P})$. The orbit $H\Lambda$ is closed if and only if $\mathcal{P}$ is an algebra partition and is of finite volume if and only if $\mathcal{P}$ is a field partition.*

Theorem 4.2 is a compressed version of Theorem 4.8, which is the main result of this section. Theorem 4.8 will be stated and proved below after some more preparations. As will be seen in Corollary 4.6, the question of whether or not a partition $\mathcal{P}$ is an algebra partition has to do with the dimension of the corresponding associated algebra. Note that the elements of $\mathcal{A}_\Lambda(\mathcal{P})$ are simply the rational matrices in $D(\mathcal{P})$ with respect to the rational structure induced by Λ. The associated algebra of a lattice Λ is a commutative algebra over $\mathbb{Q}$. It is finite-dimensional because it can be conjugated into $\mathrm{Mat}_{n\times n}(\mathbb{Q})$.

Proposition 4.3. *For any lattice Λ and any partition $\mathcal{P}$ we have that $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}) = \dim_{\mathbb{R}} (\mathcal{A}_\Lambda(\mathcal{P}) \otimes_{\mathbb{Q}} \mathbb{R})$. In particular, $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}) \le |\mathcal{P}|$ with equality if and only if $D(\mathcal{P})$ is a subspace of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ which is defined over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-structure induced by Λ.*

Proof. It is well known that for any number field F one has the equality $\dim_{\mathbb{Q}} F = \dim_{\mathbb{R}} (F \otimes_{\mathbb{Q}} \mathbb{R})$. Since $\mathcal{A}_\Lambda(\mathcal{P})$ is a semisimple algebra, the Artin-Wedderburn theorem implies that it is isomorphic to a direct sum of number fields and the first part of the Proposition follows. The dimension bound follows from the natural inclusion $\mathcal{A}_\Lambda(\mathcal{P}) \otimes_{\mathbb{Q}} \mathbb{R} \subset D(\mathcal{P})$. Finally, as noted above, with respect to the $\mathbb{Q}$-structure induced by Λ, $\mathcal{A}_\Lambda(\mathcal{P})$ consists of exactly the rational points of $D(\mathcal{P})$ and therefore, by the first part, $D(\mathcal{P})$ has a basis consisting of rational matrices if and only if $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}) = \dim_{\mathbb{R}} D(\mathcal{P})$. □

Since matrices in $D(\mathcal{P})$ commute with matrices in $H(\mathcal{P})$, we have:

Proposition 4.4. *The assignment $\Lambda \mapsto \mathcal{A}_\Lambda(\mathcal{P})$ is constant along $H(\mathcal{P})$-orbits. Therefore, $\mathcal{A}_\Lambda(\mathcal{P})$ is an invariant attached to the orbit $H(\mathcal{P})\Lambda$.*

□

Theorem 4.5. *Let $\Lambda \in \mathcal{L}_n$ be a lattice and $B \subset \mathcal{A}_\Lambda$ a subalgebra. Then there is an isomorphism of $\mathbb{Q}$-algebras $\varphi : \bigoplus_{j=1}^r F_j \to B$, where the F_j's are totally real number fields of degrees $d_j \overset{\mathrm{def}}{=} \deg(F_j/\mathbb{Q})$ such that*

$$\mathcal{P}_B = \bigsqcup_{\substack{j=1,\dots,r\\ k=1,\dots,d_j}} I_{j,k} \tag{10}$$

where

1. *For each j, the number $s_j \overset{\mathrm{def}}{=} |I_{j,k}|$ is independent of k.*
2. *For each j,k, there is a field embedding $\sigma : F_j \to \mathbb{R}$ such that for all $i \in I_{j,k}$, $\sigma = p_i \circ \varphi|_{F_j}$. Moreover any field embedding of F_j appears for some choice of k.*

Before proving Theorem 4.5 we deduce a characterization of the partitions that Theorem 4.2 may be applied to.

Corollary 4.6. *Let $\Lambda \in \mathcal{L}_n$ be given. A partition $\mathcal{P}$ is an algebra partition for Λ if and only if $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}) = |\mathcal{P}|$ and in that case,*

$$\mathcal{P} = \mathcal{P}_B, \quad \text{where } B = \mathcal{A}_\Lambda(\mathcal{P}).$$

Proof of Corollary 4.6. Suppose $\mathcal{P}$ is an algebra partition for Λ; that is, there exists a subalgebra $B \subset \mathcal{A}_\Lambda$ such that $\mathcal{P} = \mathcal{P}_B$. It follows from the definition that $\mathcal{A}_\Lambda(\mathcal{P}_B) \supset B$ so by Proposition 4.3, in order to conclude that $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}_B) = |\mathcal{P}_B|$, it is enough to show that $\dim_{\mathbb{Q}} B = |\mathcal{P}_B|$. The latter statement follows from the description of $\mathcal{P}_B$ given in Theorem 4.5.

In the other direction, assume that $\mathcal{P}$ satisfies $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}) = |\mathcal{P}|$ and denote $B = \mathcal{A}_\Lambda(\mathcal{P})$. Then it follows from the definitions that $\mathcal{P}$ refines $\mathcal{P}_B$. Again, by Theorem 4.5 we deduce that $\dim_{\mathbb{Q}} B = |\mathcal{P}_B|$ and so the partitions $\mathcal{P}, \mathcal{P}_B$ are equal. □

For the proof of Theorem 4.5 we will require the following well-known fact (for which we were unable to find a reference).

Lemma 4.7. *Let F be a number field of degree d over $\mathbb{Q}$, let $\sigma_i : F \to \mathbb{C}$, $i = 1, \dots, d$ be its distinct embeddings in $\mathbb{C}$, and let $k_1, \dots, k_d \in \mathbb{Z}$ be such that for all $x \in F$, $\prod_1^d \sigma_i(x)^{k_i} \in \mathbb{Q}$. Then all the k_i's are equal.*

Proof. Assume by contradiction that not all the k_i's are equal. Without loss of generality we may assume that k_1 is the minimal one and that $k_1 < k_2$ say. Since the norm map $N(x) = \prod \sigma_i(x)$ has its values in $\mathbb{Q}$, we may divide through by $N(x)^{k_1}$ to assume that $k_1 = 0 < k_2$ and all the other k_i's are non-negative. Choose a basis $\alpha_1, \dots, \alpha_d$ of F over $\mathbb{Q}$, and denote by φ the polynomial

$$(X_1, \dots, X_d) \mapsto \prod_{i=1}^{d} \sigma_i \left(\sum_j \alpha_j X_j \right)^{k_i},$$

which we can simplify as

$$\varphi(\vec{X}) = \prod_{i=1}^{d} L_i(\vec{X})^{k_i}, \quad \text{where } L_i(\vec{X}) \stackrel{\text{def}}{=} \sum_j \sigma_i(\alpha_j) X_j.$$

The L_i are linearly independent linear functionals. Thus the zero set of φ is the union of the kernels of those L_i for which $k_i \neq 0$; in particular φ is identically zero on $\ker(L_2)$ but not on $\ker(L_1)$.

Now let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism such that $\sigma_1 = \sigma \circ \sigma_2$. Then for each $\vec{X} \in \mathbb{Q}^d$ we have $\varphi(X) \in \mathbb{Q}$, hence $\sigma \circ \varphi(\vec{X}) = \varphi(\vec{X})$, and since $\mathbb{Q}^d$ is Zariski dense in $\mathbb{C}^d$, this implies that $\sigma \circ \varphi$ and φ are identical as polynomial maps. On the other hand $\sigma_i \mapsto \sigma \circ \sigma_i$ is a permutation $\sigma_i \mapsto \sigma_{\pi(i)}$ with $\pi(2) = 1$. This means that $\varphi(X)$ can also be written as $\prod_1^d L_{\pi(i)}(X)^{k_i}$, and so φ is identically zero on $\ker(L_1)$ — a contradiction. $\square$

Proof of Theorem 4.5. The fact that B is isomorphic to a direct sum of number fields is a consequence of the Artin-Wedderburn Theorem and follows from the fact that it is a finite dimensional semisimple $\mathbb{Q}$-algebra (see also [T, Prop. 3.1]). So we have an abstract isomorphism $\varphi : \bigoplus_{j=1}^r F_j \to B$, where the F_j's are number fields, and for each i, consider the restriction of p_i to B, which we continue to denote by p_i. Since the diagonal embedding of $\mathbb{Q}$ as scalar matrices is a subalgebra of B, each p_i is non-zero. For each j, let 1_j denote the image of $1 \in F_j$ in B. Then for $j \neq j'$ we have $1_j \cdot 1_{j'} = 0$. Since $\mathbb{R}$ has no zero-divisors, for each i there is a unique j such that $p_i(1_j) \neq 0$. This implies that $p_i \circ \varphi|_{F_j} : F_j \to \mathbb{R}$ is a non-zero map that respects addition and multiplication. It follows that it is a real field embedding.

To prove assertions (1),(2) it remains to show that for each j, and each field embedding $\sigma : F_j \to \mathbb{C}$, the number of indices i for which $\sigma = p_i \circ \varphi$ is a nonzero number independent of σ. To this end, for each $x \in F_j$ let

$$\psi(x) \stackrel{\text{def}}{=} \varphi(x) + \sum_{j' \neq j} 1_{j'} \in B \subset \mathcal{A}_\Lambda. \tag{11}$$

This is a diagonal matrix whose i-th entry is 1 if $p_i \circ \varphi|_{F_j}$ is zero, and is $p_i \circ \varphi(x)$ otherwise. In particular, $\det \psi(x)$ is the product of the numbers $p_i \circ \varphi(x)$, taken over the indices i for which $p_i \circ \varphi|_{F_j}$ is not zero, and is a rational number, since V_Λ — on which $\psi(x)$ acts — is a $\mathbb{Q}$-vector space. So the claim follows from Lemma 4.7.

$\square$

4.3. **Recognizing intermediate lattices.** We are now in a position to prove the main result of this section.

Theorem 4.8. *Let $\Lambda \in \mathcal{L}_n$, let $\mathcal{P}$ be a partition, and let $H = H(\mathcal{P})$, $Z = Z(\mathcal{P})$ be the corresponding groups. The following are equivalent:*

(1) *$H\Lambda$ is closed in $\mathcal{L}_n$.*
(2) *H is defined over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-structure induced by Λ.*
(3) *Z is defined over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-structure induced by Λ.*

(4) $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}) = |\mathcal{P}|$ *(that is, $\mathcal{P}$ is an algebra partition for Λ, by Corollary 4.6).*

Moreover, the orbit $H\Lambda$ is of finite volume if and only if the associated algebra $\mathcal{A}_\Lambda(\mathcal{P})$ is a field of degree $|\mathcal{P}|$ over $\mathbb{Q}$.

Proof of the first part of Theorem 4.8. Recall the notation of Proposition 3.4: $Z = Z(\mathcal{P}), S = S(\mathcal{P})$, $H_\Lambda = \mathrm{Stab}_H(\Lambda)$, $H_0 = \mathrm{Zcl}(H_\Lambda)$, $H_1 = H_0^\circ$, where Zcl stands for Zariski closure. Throughout the proof we will only refer to the $\mathbb{Q}$-structure induced by Λ.
(1) $\implies$ (3): As we saw in the proof of Proposition 3.4, H_0 is a $\mathbb{Q}$-algebraic group, contains H_1 as a finite-index subgroup, and moreover $S \subset H_1 = Z_a \cdot S \subset H$ and so H_0 is reductive. Let G_1 denote the centralizer of H_1 in G, which is again a reductive $\mathbb{Q}$-algebraic group containing Z. By [Rag, Proposition 10.15] this implies that the orbit $G_1\Lambda$ is closed and hence can be viewed as the quotient of a reductive $\mathbb{Q}$-group by its integral points. We now claim that $Z \subset G_1$ is a maximal $\mathbb{R}$-diagonalizable group and that $Z\Lambda$ is a closed orbit. Assuming this, applying [TW, Theorem 1.1], we find that Z is a $\mathbb{Q}$-subgroup of G_1, and hence a $\mathbb{Q}$-subgroup of G. Thus the claim implies (3).

From the definitions $Z = A \cap G_1$. As G_1 is normalized by A we deduce that Z is a maximal $\mathbb{R}$-diagonalizable subgroup of G_1. Since both orbits $H\Lambda$, $G_1\Lambda$ are closed, by [Sh, Lemma 2.2] so is $(H \cap G_1)\Lambda$ and since Z is of finite index in $H \cap G_1$, $Z\Lambda$ is closed as well.

(3) $\implies$ (2): As H is the connected component of the identity in the centralizer of Z in G, if Z is a $\mathbb{Q}$-algebraic group, so is H.

(2) $\implies$ (1): Since H is reductive and defined over $\mathbb{Q}$, it follows from [Rag, Proposition 10.15] that $H\Lambda$ is closed in $\mathcal{L}_n$.

(4) $\implies$ (3): It follows from Proposition 4.3 that if $\dim_{\mathbb{Q}} \mathcal{A}_\Lambda(\mathcal{P}) = |\mathcal{P}|$ then $D(\mathcal{P})$ is defined over $\mathbb{Q}$. As Z is of finite index in $D(\mathcal{P}) \cap G$ we conclude that Z is defined over $\mathbb{Q}$.

(3) $\implies$ (4): If Z is defined over $\mathbb{Q}$, it contains a Zariski dense subset of Λ-rational matrices. These are by definition elements of $\mathcal{A}_\Lambda(\mathcal{P})$ so we conclude that the dimension of the real vector space $\mathrm{Zcl}(\mathcal{A}_\Lambda(\mathcal{P}))$ is at least the dimension of Z, which is $|\mathcal{P}| - 1$. On the other hand, the line of scalar matrices is always in this space (regardless of Λ) and so the dimension is $|\mathcal{P}|$ as desired. □

In order to complete the proof of Theorem 4.8 we will need the following Proposition which relates the structure of $\mathcal{A}_\Lambda$ with the structure of the stabilizer of Λ in $Z(\mathcal{P})$.

Proposition 4.9. *Let $\Lambda \in \mathcal{L}_n$, $\mathcal{P}$ a partition, and $Z = Z(\mathcal{P})$. Let $F_1, \ldots, F_r$ and φ be as in Theorem 4.5 applied to $B = \mathcal{A}_\Lambda(\mathcal{P})$. Then* *the group*

$$Z_\Lambda \stackrel{\mathrm{def}}{=} \{a \in Z : a\Lambda = \Lambda\}$$

is contained as a finite index subgroup in $\varphi\left(\prod_1^r \mathcal{O}_j^\times\right)$*, where* $\mathcal{O}_j^\times$ *is the (multiplicative) group of units of the ring of integers of* F_j*.*

Proof. We first prove the inclusion $Z_\Lambda \subset \varphi\left(\prod_1^r \mathcal{O}_j^\times\right)$. Suppose $a \in Z_\Lambda \subset \mathcal{A}_\Lambda$. In light of Theorem 4.5 there are $x_j \in F_j$ such that for each $i \in I_{j,k}$, $p_i(a) = \sigma_k(x_j)$, where the σ_k are the distinct field embeddings of F_j. We need to show that each x_j is a unit in the ring of integers of F_j. Let M be a matrix representing the action of a, with respect to a basis of $\mathbb{R}^n$ which generates Λ. Since a preserves Λ, M has integral entries, and has x_j as an eigenvalue. Thus x_j is a root of the characteristic polynomial of M which is a monic polynomial over the integers, and so is an algebraic integer. As the same argument applies to a^{-1}, x_j^{-1}, M^{-1} we conclude that x_j is a unit.

We now show that Z_Λ is of finite index in this inclusion. For a fixed j, let $d = d_j \stackrel{\mathrm{def}}{=} \deg(F_j/\mathbb{Q})$. By Dirichlet's theorem, $\mathcal{O}_j^\times$ contains $d-1$ multiplicatively independent elements $\alpha_1, \ldots, \alpha_{d-1}$, so it suffices to show that a finite power of each M_i fixes Λ, where $M_i = \psi(\alpha_i) \in \mathcal{A}_\Lambda(\mathcal{P})$, where ψ is as in (11) above.

To this end, fix i and write $\alpha = \alpha_i$, $M = M_i$, and note that by (11) and Theorem 4.5 the characteristic polynomial $p_M(X)$ of M is of the form

$$p_M(X) = [m_\alpha(X)]^{b_1}[X-1]^{b_2},$$

where $m_\alpha(X)$ denotes the minimal polynomial of α and b_1, b_2 are non-negative integers. In particular, $p_M(X)$ has coefficients in $\mathbb{Z}$ and degree n. This implies that the additive group $\widetilde{\Lambda}$ generated by $\bigcup_{k=0}^{n-1} M^k\Lambda$ is M-invariant. Representing M with respect to a basis of V_Λ contained in Λ, we see that M has rational coefficients, and in particular $\widetilde{\Lambda}$ is discrete, and therefore is a lattice in $\mathbb{R}^n$. Since $\widetilde{\Lambda}$ contains Λ, it must contain it as a subgroup of finite index, and since $\det M = \pm 1$, the index is preserved by the action of M. Since $\widetilde{\Lambda}$ contains only finitely many subgroups of a given index, there is a power of M preserving Λ, as required. □

The following Corollary verifies the last statement of Theorem 4.8.

Corollary 4.10. *Let* $H\Lambda$ *be a closed orbit of the block group* $H = H(\mathcal{P})$ *and let* $Z = Z_s \cdot Z_a$ *be the decomposition of* $Z = Z(\mathcal{P})$ *given in Proposition 3.4. Then, the number of summands in the decomposition of the associated algebra as a direct sum of number fields* $\mathcal{A}_\Lambda(\mathcal{P}) \cong \oplus_1^r F_j$ *satisfies* $\dim Z_s = r - 1$*.*

In particular, the following are equivalent:

(i) $H\Lambda$ *is of finite volume.*
(ii) $\mathcal{A}_\Lambda(\mathcal{P})$ *is a field.*
(iii) $Z\Lambda$ *is compact.*

Proof. Denote $\deg(F_j/\mathbb{Q}) = d_j$. By the part of Theorem 4.8 already established, $\dim Z = \dim Z_s + \dim Z_a = |\mathcal{P}| - 1 = \sum_{j=1}^r d_j - 1$. By combining Proposition 4.9 and part (2) of Remark 3.7 we conclude that $\dim Z_a = \sum_{j=1}^r (d_j - 1) = \sum_{j=1}^r d_j - r$. Combining these two equalities we conclude that $r = \dim(Z_s) - 1$ as desired.

Finally, by Proposition 3.4 we know that the closed orbit $H\Lambda$ is of finite volume if and only if $\dim Z_s = 0$, which, by the above reasoning, implies the equivalence of (i) and (ii) and shows that they imply (iii). For the reverse implication (iii) $\implies$ (i), note that if $Z\Lambda$ is compact then by Proposition 3.4(2), Z_s must be trivial and therefore by Proposition 3.4(1), $H\Lambda$ is of finite volume.

□

5. Consequences and examples

Proposition 3.2 and Theorem 4.8 furnish a link between the algebraic properties of intermediate lattices and the structure of their orbits under block groups. This sheds light on all possible A-invariant homogeneous spaces. In this section we collect results in this direction, and conclude with some examples.

Corollary 5.1. *For any lattice $\Lambda \in \mathcal{L}_n$, the map $B \mapsto H(\mathcal{P}_B)$ is a bijective correspondence between the subalgebras of $\mathcal{A}_\Lambda$, and the block groups H for which $H\Lambda$ is a closed orbit. Under this bijection subfields of $\mathcal{A}_\Lambda$ correspond to finite volume orbits. The bijection is order-reversing for the orderings of the corresponding sets by inclusion.*

Proof. This follows from Theorem 4.2 and Corollary 4.6. □

In Corollary 5.1, the trivial algebra $\mathbb{Q}$ corresponds to the block group $H = G$ (the group with one block). Recalling Definition 1.5, we obtain:

Corollary 5.2. *A lattice $\Lambda \in \mathcal{L}_n$ is intermediate (resp. intermediate of finite volume type) if and only if the associated algebra $\mathcal{A}_\Lambda$ is nontrivial (resp. contains a subfield other than $\mathbb{Q}$).*

□

In the other extreme, for $A = H(\mathcal{P}_0)$ we have the following Corollary which explains the terminology in Definition 1.6.

Corollary 5.3. *A lattice $\Lambda \in \mathcal{L}_n$ is an algebra lattice (resp. a number field lattice) if and only if the associated algebra is n-dimensional over $\mathbb{Q}$ (resp. is a field of degree n over $\mathbb{Q}$).*

The following Corollary recovers a result from [LW]. Note that by Theorem 4.5 any field partition must be an equiblock partition; that is, a partition into blocks of equal size.

Corollary 5.4 (See §6 in [LW])**.** *If $H = H(\mathcal{P})$ has a finite volume orbit in $\mathcal{L}_n$ then H is necessarily an equiblock group.*

□

We now summarize the relation between subalgebras and partitions. Given a lattice Λ we have defined two maps

$$\begin{tikzcd} \{\text{partitions } \mathcal{P}\} \arrow[r, bend left, "\mathcal{P} \mapsto \mathcal{A}_\Lambda(\mathcal{P})"] & \{\text{subalgebras of } \mathcal{A}_\Lambda\} \arrow[l, bend left, "\mathcal{P}_B \leftarrow B"] \end{tikzcd} \tag{12}$$

The image of the RHS in the LHS of (12) is the collection of algebra partitions which are those of dynamical interest, in light of Corollary 5.1.

Proposition 5.5. *Let Λ be a lattice.*

(1) *Both maps in (12) respect the partial orderings of refinement on the LHS and inclusion on the RHS.*
(2) *For any subalgebra $B \subset \mathcal{A}_\Lambda$ we have that $B = \mathcal{A}_\Lambda(\mathcal{P}_B)$; that is, going from the RHS to the LHS and back in (12) is the identity map.*
(3) *In the other direction, for any partition $\mathcal{P}$, if $B = \mathcal{A}_\Lambda(\mathcal{P})$ then $\mathcal{P}_B$ is the finest algebra partition which is coarser than $\mathcal{P}$.*

We will not be using Proposition 5.5 and its proof is left to the reader. □

5.1. **Density properties.** It is a well-known result of Prasad and Raghunathan [PrRa], based on earlier work of Mostow, that the set of compact A-orbits is dense in any fixed finite volume orbit $H\Lambda$ of a block group H. Our results imply the following related result:

Proposition 5.6. *Let $H_1 \subsetneq H_2$ be two equiblock subgroups of G and let Λ_0 be an intermediate lattice such that both orbits $H_i\Lambda_0$ are homogeneous and of finite volume. Let $\mathcal{P}_i$ denote the partition satisfying*

$H_i = H(\mathcal{P}_i)$ and let $K = \mathcal{A}_{\Lambda_0}(\mathcal{P}_1)$ be the subfield of the associated algebra to Λ_0 corresponding to $\mathcal{P}_1$ by Corollary 5.1. Then the set

$$\{\Lambda \in H_2\Lambda_0 : H_1\Lambda \text{ is a homogeneous subspace and } K \cong \mathcal{A}_\Lambda(\mathcal{P}_K)\}$$

is dense in $H_2\Lambda_0$.

Proof. Let $Z = Z(\mathcal{P}_1)$. By Corollary 4.10, $Z\Lambda_0$ is compact. Let Γ' be the subgroup of H_2 fixing Λ_0, which is an arithmetic lattice in H_2. Let $H_2(\mathbb{Q})$ denote the elements $q \in H_2$ which are rational with respect to the corresponding $\mathbb{Q}$-structure, a dense subgroup of H_2 (see [LW, Prop. 3.4] for more details). For each $q \in H_2(\mathbb{Q})$, $q\Gamma' q^{-1}$ is commensurable with Γ', so the orbits $Z\Lambda_0$ and $Zq\Lambda_0$ share a common finite cover. This implies that each $Zq\Lambda_0$ is also compact. Another application of Corollary 4.10 shows that $H_1 q\Lambda_0$ is also a homogeneous subset of finite volume. Moreover, for each $q \in H_2(\mathbb{Q})$, $V_{\Lambda_0} = V_{q\Lambda_0}$. This implies that $\mathcal{A}_{\Lambda_0}(\mathcal{P}_K) = \mathcal{A}_{q\Lambda_0}(\mathcal{P}_K)$. So the set of lattices $\{q\Lambda_0 : q \in H_2(\mathbb{Q})\}$ has the desired properties. ☐

5.2. **Constructing intermediate lattices.** Let $B = \oplus_1^r F_j$ be an n-dimensional $\mathbb{Q}$-algebra where the F_j's are totally real number fields of degrees d_j over $\mathbb{Q}$ respectively; so $\dim_{\mathbb{Q}} B = \sum_1^r d_j = n$. Let $\sigma_i : B \to \mathbb{R}, i = 1 \dots n$ be some enumeration of the n distinct homomorphisms of B into the reals. More concretely, if we denote by $\tau_{jk} : F_j \to \mathbb{R},\ k = 1, \dots, d_j$ the various field embeddings of F_j into the reals, and view each τ_{jk} as a homomorphism from B to $\mathbb{R}$, then $\sigma_1, \dots, \sigma_n$ is some enumeration of the $\tau_{jk},\ j = 1, \dots, r, k = 1, \dots, d_j$. Let $v : B \to \mathbb{R}^n$ be the map

$$\alpha \mapsto v(\alpha) \stackrel{\text{def}}{=} (\sigma_1(\alpha), \dots, \sigma_n(\alpha)) \in \mathbb{R}^n. \tag{13}$$

Let L be an additive subgroup of B of rank n. As $B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^n$, the group $\{v(\alpha) : \alpha \in L\}$ is a lattice in $\mathbb{R}^n$. Let

$$\Lambda_L \stackrel{\text{def}}{=} c_L \{v(\alpha) : \alpha \in L\}, \tag{14}$$

where c_L is chosen so that Λ_L has covolume 1 and so belongs to $\mathcal{L}_n$. Lattices arising in this way have been studied by many authors (mainly in the case where B is a field), see e.g. [GL, Chap. 1] or [PR, p. 54]. We refer to below to lattices of the form Λ_L as *lattices arising via* (14).

The following proposition gives an explicit construction of all algebra lattices.

Proposition 5.7. (1) *Let Λ_L be a lattice arising via (14) with L a rank-n subgroup of the n-dimensional $\mathbb{Q}$-algebra B as above. Then the associated algebra $\mathcal{A}_{\Lambda_L}$ is isomorphic to B. In particular, the orbit $A\Lambda_L$ is closed and so consists of algebra lattices.*

(2) *If Λ is an algebra lattice, then there is a lattice Λ_L arising via (14) with $\Lambda \in A\Lambda_L$.*

Proof. (1) The map $v : B \to \mathbb{R}^n$ in (13) is $\mathbb{Q}$-linear and also respects multiplication in the sense that

$$v(\alpha \cdot \beta) = \mathrm{diag}(\sigma_1(\alpha), \ldots, \sigma_n(\alpha)) \cdot v(\beta).$$

This means that $V_{\Lambda_L} = c_L v(B)$, and moreover that the map $\alpha \mapsto \mathrm{diag}(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$ is an embedding of B into $\mathcal{A}_{\Lambda_L}$. As B is n-dimensional and $\mathcal{A}_{\Lambda_L}$ is at most n-dimensional, the above map is an isomorphism between B and $\mathcal{A}_{\Lambda_L}$. By Corollary 5.3, Λ_L is an algebra lattice as desired.
(2) Let $B = \mathcal{A}_\Lambda$. By Corollary 5.3, B is n-dimensional. Choose a vector $w \in \Lambda$ all of whose coordinates are positive and consider the map $\psi : B \to V_\Lambda$ given by $a \mapsto a \cdot w$. This map is clearly $\mathbb{Q}$-linear and injective. As V_Λ is n-dimensional as well, it must be an isomorphism of $\mathbb{Q}$-vector spaces. Let $L \overset{\mathrm{def}}{=} \psi^{-1}(\Lambda) \subset B$. By Theorem 4.5, by projecting B to the diagonal coordinates we obtain an ordering of all the various homomorphisms of B into $\mathbb{R}$. This way we obtain a map $v : B \to \mathbb{R}^n$ as in (13), and the lattice Λ_L which arises via (14) satisfies $\Lambda_L = a \cdot \Lambda$. Here a is the diagonal matrix obtained by rescaling the diagonal matrix $\mathrm{diag}(w_i)$ to have determinant 1. Indeed $a \in A$ as we chose w so that all of its coordinates are positive. $\square$

The following Corollary is a refined version of Proposition 3.4 (when applied to a closed A-orbit) in conjunction with the concrete description of algebra lattices given in Proposition 5.7 above.

Corollary 5.8. *Suppose Λ is an algebra lattice. Then there is a decomposition $A = T_s \times T_a$ and a direct sum decomposition $\mathbb{R}^n = \bigoplus_1^r V_j$ such that the following hold:*

(i) *Each V_j is spanned by some of the standard basis vectors.*
(ii) *T_a is the group of diagonal (with respect to the standard basis) matrices whose restriction to each V_j has determinant 1.*
(iii) *T_s is the group of linear transformations which act on each V_j by a homothety, preserving Lebesgue measure on $\mathbb{R}^n$. In particular $\dim T_s = r - 1$.*
(iv) *$T_s x$ is divergent and $T_a x$ is compact.*
(v) *Setting $\Lambda_j \overset{\mathrm{def}}{=} V_j \cap \Lambda$, each Λ_j is a lattice in V_j, so that $\bigoplus \Lambda_j$ is of finite index in Λ.*

Proof. Since our required conclusions are invariant when replacing Λ with $a\Lambda$ for $a \in A$, by Proposition 5.7 we can assume that $\Lambda = \Lambda_L$

arises via (14). With the notation above, we set V_j to be the span of the standard basis vectors $\mathbf{e}_i$ for which $\sigma_i = \tau_{jk}$ for $k \in \{1, \dots, d_j\}$, so (i) holds. Let $\mathcal{P} = \mathcal{P}_0$, so that $A = H(\mathcal{P}) = Z(\mathcal{P})$. Let T_a be the Zarizki closure of the stabilizer A_Λ. By Proposition 4.9 T_a is contained in the group $\widetilde{T}$ of diagonal matrices whose restriction to each V_i has determinant one. A dimension count using the Dirichlet unit theorem now implies that $T_a = \widetilde{T}$, establishing (ii). Recalling Proposition 3.4 (and its proof) we have that $T_a\Lambda$ is compact and that if T_s is any choice of a direct complement of T_a in A, then $T_s\Lambda$ is divergent. Defining T_s by (iii), one obtains (iv). Since L generates B over $\mathbb{Q}$, and since each F_j is the pre-image of V_j under the map (13) and is a $\mathbb{Q}$-subspace of B, (v) holds. $\square$

The following Proposition gives an explicit construction of all homogeneous A-invariant spaces in $\mathcal{L}_n$ (or equivalently, of all intermediate lattices). It shows that each such homogeneous space $H(\mathcal{P})\Lambda$ contains an algebra lattice Λ_L arising via (14). Moreover, by Corollary 5.1, the partition $\mathcal{P}$ must be the algebra partition that corresponds to the subalgebra of $\mathcal{A}_{\Lambda_L}$ that we associate to the homogeneous space.

Proposition 5.9. *Let $H\Lambda$ be a closed orbit for the group $H = H(\mathcal{P})$. Then there exists a lattice Λ_L arising via (14) such that $\Lambda \in H\Lambda_L$. If $H\Lambda$ is of finite volume then Λ_L can be taken to be a number field lattice.*

Proof. By Proposition 3.4 (and its notation), we may present H as a direct product $H = Z_s \cdot Z_a \cdot S$. Since S is a semisimple group defined over $\mathbb{Q}$, with respect to the $\mathbb{Q}$-structure induced by Λ, Proposition 2.4 implies that the orbit $S\Lambda$ is of finite volume. Let $A_0 \stackrel{\text{def}}{=} S \cap A$. By the theorem of Prasad and Raghunathan [PrRa] the finite volume orbit $S\Lambda$ contains a lattice Λ' with a compact A_0-orbit. By Remark 3.7(2) we have that $Z_a\Lambda'$ is compact as well and since Z_a commutes with A_0, the orbit $Z_aA_0\Lambda'$ is compact. Applying part (3) of Proposition 3.4 we deduce that the orbit $Z_sZ_a\,A_0\Lambda'$ is closed but as $A = Z_sZ_a\,A_0$ we conclude that Λ' is an algebra lattice. Similarly, when $H\Lambda$ is of finite volume then $A\Lambda'$ is compact so Λ' is a number field lattice. By Proposition 5.7 we conclude that we may assume without loss of generality that $\Lambda' = \Lambda_L$ is a lattice arising via (14). $\square$

5.3. **Indecomposable lattices.**

Proposition 5.10. *Let μ be a finite A-invariant homogeneous measure. Then μ-a.e. Λ is indecomposable. In particular $\kappa_\mu \in \widehat{\mathbf{MG}}_n$ (where κ_μ is as in (5)).*

Proof. A decomposable lattice $\Lambda = \Lambda_1 \oplus \Lambda_2$ has nonzero vectors with zero entries (namely those nonzero vectors in each Λ_i, as embedded in Λ). If the set of decomposable lattices had positive μ-measure, then for some index i_0, there would be a set of positive measure of lattices Λ containing a vector whose i_0-th coordinate vanishes. Assume to simplify notation that $i_0 = 1$, then by Proposition 2.2, such lattices have a divergent trajectory under the one parameter subgroup $a_t = \mathrm{diag}(e^{(n-1)t}, e^{-t}, \ldots, e^{-t})$ as $t \to \infty$. This contradicts the Poincaré recurrence theorem, which asserts that with respect to an invariant probability measure for an $\mathbb{R}$-action, almost every point x returns to any neighborhood of x along an unbounded infinite subsequence. $\square$

5.4. **Examples.**

Example 5.11. Let $\Lambda = \mathbb{Z}^n$, that is, Λ arises via (14) from the n-dimensional $\mathbb{Q}$-algebra $B = \mathbb{Q}^n$ and so $\mathcal{A}_\Lambda \cong \mathbb{Q}^n$. Moreover, for any partition $\mathcal{P}$, the subalgebra $\mathcal{A}_\Lambda(\mathcal{P})$ consists of all diagonal matrices with rational diagonal entries that are constant in each block of $\mathcal{P}$ and so is a $|\mathcal{P}|$-dimensional subalgebra of $\mathcal{A}_\Lambda$. By Corollary 4.6 any partition $\mathcal{P}$ is an algebra partition; hence by Corollary 5.1, the orbit $H(\mathcal{P})\Lambda$ is closed for any block group $H(\mathcal{P})$. Moreover, as $\mathbb{Q}^n$ does not have any subfields other than $\mathbb{Q}$, all orbits $H(\mathcal{P})\Lambda$ are of infinite volume, apart from the orbit $\mathcal{L}_n$ which is obtained by choosing the trivial partition that contains only one block, corresponding to the subfield $\mathbb{Q}$.

Example 5.12. Let $B = F_1 \oplus F_2$ be a 4-dimensional $\mathbb{Q}$-algebra where $F_1 = F_2 = \mathbb{Q}(\sqrt{2})$. Denote by $x \mapsto x'$ the nontrivial automorphism of $\mathbb{Q}(\sqrt{2})$. Let $L = \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2}$ and define $\Lambda = \Lambda_L$ to be the lattice defined by (14) where $v(x, y) = (x, x', y, y')$. Then

$$\Lambda_L = c_L\{(x, x', y, y') : x, y \in \mathcal{O}_{\mathbb{Q}(\sqrt{2})}\}$$

is an algebra lattice with $F_1 \oplus F_2 \cong \mathcal{A}_\Lambda$. The isomorphism is given by the map $(x, y) \mapsto \mathrm{diag}(x, x', y, y')$. It is not hard to write down a table of all subalgebras of $\mathcal{A}_\Lambda$ and work out the corresponding algebra partitions. This gives us a classification of all closed orbits of block groups through Λ. For example if we take $B_1 = \{\mathrm{diag}(x, x, y, y) : x, y \in \mathbb{Q}\} \cong \mathbb{Q} \oplus \mathbb{Q}$ we obtain the algebra partition $\mathcal{P}_1 = \{\{1, 2\}, \{3, 4\}\}$ for which (by Corollary 5.1) the orbit $H(\mathcal{P}_1)\Lambda$ is closed but of infinite volume because $\mathbb{Q} \oplus \mathbb{Q}$ is not a field. On the other hand, if we take $B_2 = \{\mathrm{diag}(x, x', x, x') : x \in \mathbb{Q}(\sqrt{2})\}$, then we obtain the algebra partition $\mathcal{P}_2 = \{\{1, 3\}, \{2, 4\}\}$ for which $H(\mathcal{P}_2)\Lambda$ is a closed orbit of finite volume as $B_2 \cong \mathbb{Q}(\sqrt{2})$ is a field.

6. Strict inequalities among the κ_μ

We begin with a definition. Let $\varphi : \oplus_{j=1}^{r_2} F_j^{(2)} \hookrightarrow \oplus_{j=1}^{r_1} F_j^{(1)}$ be an embedding of $\mathbb{Q}$-algebras. We say that φ is *essential* if the image of φ projects onto $F_j^{(1)}$ for any $1 \leq j \leq r_1$. Otherwise we refer to φ as *non-essential*.

We can now state the main result of this section, which is one of the main results of this paper.

Theorem 6.1. *Let μ_i, $i = 1, 2$ be two homogeneous A-invariant measures such that $\mathrm{supp}(\mu_1) \subsetneq \mathrm{supp}(\mu_2)$. Let $H_i = H(\mathcal{P}_i)$ and $\Lambda \in \mathcal{L}_n$ be such that $H_i\Lambda = \mathrm{supp}(\mu_i)$. If the containment $\mathcal{A}_\Lambda(\mathcal{P}_2) \subset \mathcal{A}_\Lambda(\mathcal{P}_1)$ is non-essential then $\kappa_{\mu_1} < \kappa_{\mu_2}$.*

Deduction of Theorem 1.4. If μ_1 is a finite measure then by Corollary 5.1 the associated algebra to the orbit that supports μ_1 is a field. It follows that the associated algebra to the orbit that supports μ_2 must be a field as well (because it is a subalgebra of a field) and therefore μ_2 must be a finite measure as well by another application of Corollary 5.1. By Lemma 3.3, since $H_1\Lambda = \mathrm{supp}(\mu_1) \subset \mathrm{supp}(\mu_2) = H_2\Lambda$, we have $H_1 \subset H_2$ and since the containment between the orbits is strict, we have $H_1 \subsetneq H_2$. Therefore the containment of the associated algebras is strict and so the containment is non-essential and Theorem 6.1 applies. We therefore conclude that if μ_1 is a finite measure, then $\kappa_{\mu_1} < \kappa_{\mu_2}$ and Theorem 1.4 follows. $\square$

Example 6.2. Continuing with Example 5.11, note that when we consider the inclusion of closed orbits $A\mathbb{Z}^n \subset G\mathbb{Z}^n$, the containment of the associated algebras $\mathbb{Q} \subset \oplus_1^n \mathbb{Q}$ is essential and the conclusion of Theorem 6.1 fails to hold as both of the generic constants attached to these orbits are equal to 1.

Example 6.3. Let Λ be the lattice constructed in Example 5.12. In the notation of that example we know that the orbits $A\Lambda, H(\mathcal{P}_1)\Lambda$, $H(\mathcal{P}_2)\Lambda$, $G\Lambda$ are all closed and their associated algebras are isomorphic respectively to $\mathbb{Q}(\sqrt{2}) \oplus \mathbb{Q}(\sqrt{2})$, $\mathbb{Q} \oplus \mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}$. We denote the generic values attached to these closed orbits by $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ respectively, so that $\kappa_3 = \kappa_{\mu_{\mathcal{L}_n}} = 1$. Because the inclusions $\mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$, $\mathbb{Q} \oplus \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2}) \oplus \mathbb{Q}(\sqrt{2})$ are non-essential we deduce by Theorem 6.1 that $\kappa_2 < \kappa_3$, $\kappa_0 < \kappa_1$. On the other hand, the inclusions $\mathbb{Q} \hookrightarrow \mathbb{Q} \oplus \mathbb{Q}$, $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{Q}(\sqrt{2}) \oplus \mathbb{Q}(\sqrt{2})$ are essential and so Theorem 6.1 does not tell us that the inequalities $\kappa_1 \leq \kappa_3$, $\kappa_0 \leq \kappa_2$ are strict. Indeed, it is not hard to see that $\kappa_1 = 1$ because the lattice $\mathbb{Z}^4$ belongs to the orbit $H(\mathcal{P}_1)\Lambda$ (as Λ is the direct sum of two 2-dimensional lattices and the

2 by 2 blocks of $H(\mathcal{P}_1)$ act on each of the summands separately). We do not know whether $\kappa_0 < \kappa_2$. In fact we do not know an example of a strict inequality between the κ-values when the containment of the associated algebras is essential.

The proof of Theorem 6.1 requires some preparations. Once again let $\varphi : \oplus_{j=1}^{r_2} F_j^{(2)} \hookrightarrow \oplus_{j=1}^{r_1} F_j^{(1)}$ be an embedding of $\mathbb{Q}$-algebras. We say that φ is *aligned* if $r_1 = r_2$.

Proposition 6.4. *Let $H\Lambda$ be a closed orbit of $H = H(\mathcal{P})$ and let $\Lambda_L \in H\Lambda$ be the algebra lattice constructed in Proposition 5.9. Then the inclusion $\mathcal{A}_{\Lambda_L}(\mathcal{P}) \subset \mathcal{A}_{\Lambda_L}$ is aligned.*

Proof. As both orbits $H\Lambda_L$ and $A\Lambda_L$ are closed, we may apply Proposition 3.4 to both of them and obtain decompositions of $Z(\mathcal{P})$ and A respectively. Following the proof of Proposition 5.9 we see that the split part in these decompositions may be chosen to be the same; indeed, in the notation of Proposition 5.9, if $Z = Z_s \cdot Z_a$ is the decomposition for $Z = Z(\mathcal{P})$, then we saw that $A = Z_s \cdot (Z_a \cdot (A \cap S))$ and that Λ_L was chosen so that $Z_a(A \cap S)\Lambda_L$ is compact. It now follows from Corollary 4.10 that if we present the associated algebras to the orbits $H\Lambda_L$, $A\Lambda_L$ as $\mathcal{A}_{\Lambda_L} = \oplus_1^{r_1} F_j^{(1)}$, $\mathcal{A}_{\Lambda_L}(\mathcal{P}) \cong \oplus_1^{r_2} F_j^{(2)}$, then $r_1 = r_2$. $\square$

6.1. The Kernel Lemma.

We will need some more notation.

Definition 6.5.
1. Given a block group $H = H(\mathcal{P})$ and a closed orbit $H\Lambda$ with associated algebra $\mathcal{A}_\Lambda(\mathcal{P}) \cong \oplus_{j=1}^{r} F_j$ such that $\deg(F_j/\mathbb{Q}) = d_j$, we present $\mathcal{P}$ as in (10), $\mathcal{P} = \sqcup_{j=1}^{r} \sqcup_{k=1}^{d_j} I_{j,k}$. Let
$$\widetilde{I}_j \overset{\text{def}}{=} \cup_{k=1}^{d_j} I_{jk} \quad \text{and} \quad \widetilde{\mathcal{P}} = \sqcup_{j=1}^{r} \widetilde{I}_j;$$
that is, the j-th block of $\widetilde{\mathcal{P}}$ is obtained by grouping the diagonal coordinates that correspond to embeddings of F_j.
2. Given a subset $Q \subset \{1, \ldots, n\}$, we denote by $\pi_Q : \mathbb{R}^n \to \mathbb{R}^{|Q|}$ the projection to the coordinates of the subset Q.

Lemma 6.6 (Kernel Lemma)**.** *Let $H\Lambda \subset \mathcal{L}_n$ be a closed orbit of the block group $H = H(\mathcal{P})$. Let Q_1, Q_2 be two blocks of $\mathcal{P}$ that are contained in the same block of the partition $\widetilde{\mathcal{P}}$ from Definition 6.5. Then, there is an automorphism ρ of $\mathbb{C}$ such that for any collection of vectors $v_1, \ldots, v_t \in \Lambda$, we have the following connection between kernels of the*

$n \times t$ matrices whose columns are the $\pi_{Q_j}(v_i)$'s

$$\ker\begin{pmatrix} | & & | \\ \pi_{Q_1}(v_1) & \cdots & \pi_{Q_1}(v_t) \\ | & & | \end{pmatrix} = \rho\left(\ker\begin{pmatrix} | & & | \\ \pi_{Q_2}(v_1) & \cdots & \pi_{Q_2}(v_t) \\ | & & | \end{pmatrix}\right), \tag{15}$$

where we let ρ act on $\mathbb{R}^n$ coordinate-wise.

Proof. Because of the block structure of H, the conclusion of the Lemma is independent of the lattice we choose to consider within the orbit $H\Lambda$. By Proposition 5.9 it is enough to assume that $\Lambda = \Lambda_L$ is constructed via (14) for some n-dimensional $\mathbb{Q}$-algebra B and $L \subset B$. By Proposition 5.7 we may assume that $B = \mathcal{A}_\Lambda$. Moreover, by the proof of Proposition 5.7, the map that sends a diagonal matrix in $\mathcal{A}_\Lambda$ to the vector in $\mathbb{R}^n$ whose coordinates are the diagonal entries of the matrix is a linear isomorphism[3] between $\mathcal{A}_\Lambda$ and V_Λ. This means that the statement of the Lemma translates to a statement about the associated algebra $\mathcal{A}_\Lambda$.

By Proposition 6.4, the containment $\mathcal{A}_\Lambda(\mathcal{P}) \subset \mathcal{A}_\Lambda$ is aligned. We present $\mathcal{A}_\Lambda \cong \oplus_{j=1}^r F_j$, $\mathcal{A}_\Lambda(\mathcal{P}) \cong \oplus_{j=1}^r K_j$ and note that the alignment of the inclusion of the algebras means that we may assume that for each $1 \le j \le r$ the field F_j is an extension of K_j and the inclusion $\mathcal{A}_\Lambda(\mathcal{P}) \subset \mathcal{A}_\Lambda$ is induced from the natural inclusion $\oplus_1^r K_j \subset \oplus_1^r F_j$.

By Theorem 4.5, as $\mathcal{A}_\Lambda$ is n-dimensional, the diagonal coordinates (or the coordinates of $\mathbb{R}^n$) are in one to one correspondence with the various field embeddings of the fields F_j. By the above discussion, the blocks $\widetilde{I}_j$ of $\widetilde{\mathcal{P}}$ are obtained by grouping the coordinates that correspond to each field F_j together. As $\mathcal{P}$ is the algebra partition attached to $\mathcal{A}_\Lambda(\mathcal{P})$ (see Definition 4.1 and Corollary 4.6), the blocks of $\mathcal{P}$ are then obtained by further splitting each $\widetilde{I}_j$ according to the restriction of the corresponding embedding of F_j to the subfield K_j. That is, two coordinates that correspond to embeddings of F_j that restrict to the same embedding of K_j belong to the same block. As the group of automorphisms of $\mathbb{C}$ acts transitively on the equivalence classes of embeddings of F_j with respect to the above equivalence relation, we deduce that if Q_1, Q_2 are two blocks of $\mathcal{P}$ that are contained in $\widetilde{I}_j$, then there is an automorphism of $\mathbb{C}$ such that for each $v \in V_\Lambda = \mathcal{A}_\Lambda$ we have that $\pi_{Q_1}(v) = \rho(\pi_{Q_2}(v))$. From here (15) readily follows. $\square$

6.2. Strict inequalities for κ values.

[3]The attentive reader will notice that V_Λ should be replaced with its dilation.

Lemma 6.7. *Let $H_1\Lambda \subset H_2\Lambda$ be a containment of two closed orbits where $H_i = H(\mathcal{P}_i)$, $i = 1, 2$. If the containment $\mathcal{A}_\Lambda(\mathcal{P}_2) \subset \mathcal{A}_\Lambda(\mathcal{P}_1)$ is non-essential, then there is a block of $\mathcal{P}_2$ that contains two distinct blocks of $\mathcal{P}_1$ that are contained in the same block of $\widetilde{\mathcal{P}}_1$.*

Lemma 6.7 together with the following Theorem implies the validity of Theorem 6.1. We postpone the proof of Lemma 6.7 to the end of this section.

Theorem 6.8. *Let $H_1\Lambda \subset H_2\Lambda$ be a containment of two closed orbits where $H_i = H(\mathcal{P}_i)$, $i = 1, 2$. Suppose that there is a block of $\mathcal{P}_2$ that contains two distinct blocks of $\mathcal{P}_1$ that are contained in the same block of $\widetilde{\mathcal{P}}_1$. Then $\kappa_1 < \kappa_2$.*

Proof. Assume by way of contradiction that $\kappa_1 = \kappa_2$. Without loss of generality we may assume that $\Lambda = \Lambda_{\max}$ is a lattice for which the conclusions of Theorem 3.1 are satisfied for the A-invariant set $H_1\Lambda$; that is, we assume that $\kappa(\Lambda) = \max\{\kappa(\Lambda') : \Lambda' \in H_1\Lambda\}$ and furthermore, that the symmetric cube $\mathcal{C}$ of volume $2^n\kappa_1$ is admissible for Λ. From our assumption we deduce that also $\kappa(\Lambda) = \max\{\kappa(\Lambda') : \Lambda' \in H_2\Lambda\}$. In practice, the property of Λ that will be of importance to us is that one cannot act on Λ with some $h \in H_2$ in such a way that a symmetric box of bigger volume will be admissible for $h\Lambda$.

For $1 \le i \le n$ we refer to the face of $\mathcal{C}$ that intersects the positive i-th axis as the i-th face of $\mathcal{C}$ and denote it by $\mathcal{F}_i$. By symmetry there will be no need to consider the opposite faces. We divide the rest of the argument into steps.

Step 1. We first observe that for each $1 \le i \le n$ the relative interior of $\mathcal{F}_i$ intersects Λ nontrivially. If not, we could have chosen an admissible symmetric box for Λ which strictly contains $\mathcal{C}$ and in particular, is of greater volume. We refer to the set $L \overset{\mathrm{def}}{=} \Lambda \cap \partial\mathcal{C}$ as the set of *locking points* and denote by L_i, $1 \le i \le n$ the set of points in L that belong to the relative interior of $\mathcal{F}_i$.

Step 2. Recall that for $Q \subset \{1, \dots, n\}$, $\pi_Q : \mathbb{R}^n \to \mathbb{R}^{|Q|}$ is the projection to the coordinates of Q. We observe that for each block Q of the partition $\mathcal{P}_2$ and for each $i_0 \in Q$,

$$0 \in \operatorname{conv}\{\pi_{Q\smallsetminus\{i_0\}}(v) : v \in L_{i_0}\}. \tag{16}$$

In other words, if we restrict attention to the coordinates of the block Q, then the point of intersection of $\mathcal{F}_{i_0}$ with the i_0-th axis belongs to the convex hull of locking points for the i_0-th face.

To prove (16), suppose to the contrary that 0 does not belong to $\operatorname{conv}\{\pi_{Q\smallsetminus\{i_0\}}(v) : v \in L_{i_0}\}$. Then there is a linear functional $f : \mathbb{R}^{|Q|-1} \to$

$\mathbb{R}$ which is strictly positive on $\{\pi_{Q\smallsetminus\{i_0\}}(v) : v \in L_{i_0}\}$. Define a one-parameter unipotent subgroup $\{u_t\}$ of G by the formula

$$u_t(v) = v + tf(\pi_{Q\smallsetminus\{i_0\}}(v))\mathbf{e}_{i_0}, \tag{17}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$. Since Q is a block of the partition defining H_2, we have $\{u_t\} \subset H_2$. It follows from the definition of u_t and the positivity of $f \circ \pi_{Q\smallsetminus\{i_0\}}$ on L_{i_0} that for small values of $t > 0$ the cube $\mathcal{C}$ is admissible for $u_t\Lambda$. Furthermore, $u_t\Lambda$ does not contain points in the relative interior of $\mathcal{F}_{i_0}$. Thus we may find a symmetric box that strictly contains $\mathcal{C}$ and is admissible for $u_t\Lambda$, contradicting our assumption that κ attains its maximal value on $H_2\Lambda$ at Λ.

Step 3. Now we apply the Kernel Lemma 6.6. Denote by Q a block of $\mathcal{P}_2$ that contains two distinct blocks Q_1, Q_2 of $\mathcal{P}_1$ such that both Q_i are contained in the same block of $\widetilde{\mathcal{P}}_1$, the existence of which is assumed in the statement. Let $i_0 \in Q_1$.

Let $\delta > 0$ be such that $\delta\mathbf{e}_{i_0}$ is the point of intersection of $\mathcal{F}_{i_0}$ with the i_0-th axis. Because all the points of $\mathcal{F}_{i_0}$ share the same i_0-th coordinate, namely δ, we deduce from (16) that $\pi_Q(\delta\mathbf{e}_{i_0})$ is in the convex hull of $\pi_Q(L_{i_0})$. Choose $v_1, \dots, v_t$ in L_{i_0} and a coefficients vector $\vec{\lambda} = (\lambda_1, \dots, \lambda_t)$ with $\lambda_i \geq 0$ and $\sum_1^t \lambda_i = 1$ so that $\sum \lambda_i \pi_Q(v_i) = \pi_Q(\delta\mathbf{e}_{i_0})$. In particular, as $i_0 \notin Q_2$, the vector $\vec{\lambda}$ belongs to the kernel of the $n \times t$ matrix whose columns are the $\pi_{Q_2}(v_i)$'s. Applying the Kernel Lemma 6.6 we deduce that there is an automorphism ρ of $\mathbb{C}$ so that $\rho(\vec{\lambda})$ belongs to the kernel of the corresponding matrix with columns $\pi_{Q_1}(v_i)$. As ρ is an automorphism we deduce that $\sum_1^t \rho(\lambda_i) = \rho(1) = 1$. Looking at the i_0 coordinate we deduce that $0 = \sum_1^t \rho(\lambda_i)\delta = \delta$, a contradiction.

□

Proof of Lemma 6.7. The associated algebras $\mathcal{A}_\Lambda(\mathcal{P}_1), \mathcal{A}_\Lambda(\mathcal{P}_2)$ are isomorphic to direct sums of number fields. Suppose $\mathcal{A}_\Lambda(\mathcal{P}_i) \cong \oplus_{j=1}^{r_i} F_j^{(i)}$. Then the inclusion $\mathcal{A}_\Lambda(\mathcal{P}_2) \subset \mathcal{A}_\Lambda(\mathcal{P}_1)$ induces an embedding

$$\varphi : \oplus_{j=1}^{r_2} F_j^{(2)} \hookrightarrow \oplus_{j=1}^{r_1} F_j^{(1)}.$$

For such an embedding, there exists a partition $\{1, \dots, r_1\} = \bigsqcup_{j=1}^{r_2} Q_j$ such that for each $1 \leq j \leq r_2$ there is an embedding $\varphi_j : F_j^{(2)} \hookrightarrow \oplus_{i \in Q_j} F_i^{(1)}$ so that φ takes the following form

$$\varphi((x_j)_{j=1}^{r_2}) = (\varphi_1(x_1)), \dots, \varphi_{r_2}(x_{r_2})) \in \left(\oplus_{i\in Q_1} F_i^{(1)}\right) \oplus \cdots \oplus \left(\oplus_{i \in Q_{r_2}} F_i^{(1)}\right).$$

Our assumption that the embedding φ is non-essential simply means that there exists some $1 \le j_0 \le r_2$ and $i_0 \in Q_{j_0}$ such that

$$\deg\left(F^{(1)}_{i_0}/\mathbb{Q}\right) > \deg\left(F^{(2)}_{j_0}/\mathbb{Q}\right). \tag{18}$$

Recall that by Theorem 4.5 the blocks of the partition $\mathcal{P}_2$ correspond to the various embeddings of the $F^{(2)}_j$. Choose a block Q that corresponds to one of the embeddings of $F^{(2)}_{j_0}$.

This block splits into several blocks of the finer partition $\mathcal{P}_1$ and this splitting is done in two steps. First, it splits into blocks $S_i, i \in Q_{j_0}$, that correspond to the embedding $\varphi_{j_0} : F^{(2)}_{j_0} \hookrightarrow \oplus_{i \in Q_{j_0}} F^{(1)}_i$ and then each block S_i further splits into $\deg\left(F^{(1)}_{j_0}/\mathbb{Q}\right) / \deg\left(F^{(2)}_i/\mathbb{Q}\right)$ blocks $S_{i\ell}$ which are the blocks of $\mathcal{P}_1$. Following the definitions we see that for each fixed i the blocks $S_{i\ell}$ belong to the same block of $\widetilde{\mathcal{P}}_1$; namely the one defined by $F^{(1)}_i$. Combining this with the inequality (18) concludes the proof. $\square$

7. Proofs of isolation results

We recall the main dynamical result of [LW], which is the basis of our proof of isolation results.

Theorem 7.1. *Let $n \ge 3$, let $H\Lambda$ be a finite-volume A-invariant homogeneous subspace of $\mathcal{L}_n$, corresponding via Corollary 5.1 to a subfield F of the associated algebra $\mathcal{A}_\Lambda$. Assume that $\overline{A\Lambda} = H\Lambda$ and let (Λ_k) be a sequence of lattices in $\mathcal{L}_n \smallsetminus H\Lambda$ converging to Λ. Then, after passing to a subsequence of the (Λ_k), there is a proper subfield $K \subsetneq F$ such that the set of accumulation points of the form $\lim a_k\Lambda_k$, $a_k \in A$, is equal to the finite-volume homogeneous space $H'\Lambda$, where H' is an equiblock group associated to K via Corollary 5.1.*

In particular, if F has no proper subfields other than $\mathbb{Q}$, then the set of accumulation points as above is the entire space $\mathcal{L}_n$.

Although the Theorem is not stated in this form in [LW], its statement follows from the proofs of [LW, Theorems 1.1, 1.3]. We include a proof for completeness. In the interest of brevity, along the way we will refer the reader to the required definitions of statements from [LW]. See also [ELMV, Theorem 4.8].

Proof. Given a sequence $(\Lambda_k)_k$, we denote by $\mathcal{F}((\Lambda_k)_k)$ the set of accumulation points of sequences $a_k\Lambda_k$ with $a_k \in A$. In light of the bijection described in Corollary 5.1, it suffices to prove that there is a subsequence $(\Lambda'_j) \subset (\Lambda_k)$ such that $\mathcal{F}((\Lambda'_j)_j) = H'\Lambda$ is a finite-volume orbit where H' is an equiblock group containing H as a proper subgroup. Let $x_0 \in H\Lambda$ be such that Ax_0 is compact. Since there are only finitely many equiblock groups $\widetilde{H}$ for which $\widetilde{H}x_0$ is a finite-volume closed orbit, we may pass to a subsequence (which we continue to denote by (Λ_k)), so that for each of these groups $\widetilde{H}$, one of the following three possibilities occurs:

(i) $(\Lambda_k)_k \subset \widetilde{H}x_0$.
(ii) $\widetilde{H}x_0 \not\subset \mathcal{F}((\Lambda_k)_k)$.
(iii) $\widetilde{H}x_0 \subset \overline{\mathcal{F}((\Lambda_k)_k) \smallsetminus \widetilde{H}x_0}$.

From now on we write $\mathcal{F} = \mathcal{F}((\Lambda_k)_k)$. The assumption on the sequence $(\Lambda_k)_k$ ensures that for $\widetilde{H} = H$, (iii) occurs (regardless of our choice of subsequence), and in particular $Hx_0 \subset \mathcal{F}$. We will show that $\mathcal{F} = H'x_0$ is a finite volume orbit of an equiblock group H' properly containing H, and this will imply $\mathcal{F} = H'\Lambda$ and conclude the proof.

Let $\widetilde{H}$ be an equiblock group such that $\widetilde{H}x_0$ is a finite-volume closed orbit and (iii) holds, and let $z \in \widetilde{H}x_0$ have a compact A-orbit. By assumption (passing to a subsequence) there are $a_k \in A$ such that the sequence $x_k = a_k\Lambda_k \notin \widetilde{H}z$ converges to z. Note that for each such subsequence, $\mathcal{F}((x_k)_k) \subset \mathcal{F}$. Let the groups N_{ij}, U_{ij} be defined as in [LW]. Then each N_{ij} acts ergodically on Az by [LW, Step 6.1]. By repeating verbatim the proof of [LW, Lemma 4.2], we find that there exist indices i, j (which may depend on z) such that $U_{ij} \not\subset \widetilde{H}$ and $U_{ij}z \subset \mathcal{F}$. Now by [LW, Steps 4.7, 4.8], using $\mathcal{F}$ instead of F, we find that there are indices i, j (depending only on $\widetilde{H}$) such that $U_{ij} \not\subset \widetilde{H}$ and $U_{ij}\widetilde{H}x_0 \subset \mathcal{F}$.

Let V be a unipotent subgroup of G of maximal possible dimension, which is normalized by A and such that $Vx_0 \subset \mathcal{F}$. Using Ratner's theorem, $\overline{Vx_0} = H_1x_0$ and by [LW, Proof of Theorem 1.1], $H' = AH_1$ is an equiblock group and $H'x_0 \subset \mathcal{F}$ is closed and of finite volume. Assume by contradiction that (iii) holds for H'. Then letting U_{ij} be such that $U_{ij}H'x_0 \subset \mathcal{F}$, $U_{ij} \not\subset H'$ and arguing as in [LW, Step 4.9] we obtain a contradiction to the maximality of V. This implies that H' satisfies (i), and this in turn implies $\mathcal{F} = H'x_0$. Since $Hx_0 \subset \mathcal{F}$ and the groups H, H' are connected, we have $H \subset H'$, and since H satisfies (iii), this containment is proper. $\square$

7.1. **Proofs.**

Proof of Theorem 1.9. For each subfield $F_1 \subset F$, by Corollary 5.1 there is a corresponding homogeneous subset $H\Lambda$, equipped with a homogeneous measure μ_{F_1}, and by Theorem 3.1, a corresponding $\kappa_{\mu_{F_1}}$. Let

$$\kappa' \stackrel{\text{def}}{=} \min\{\kappa_{\mu_{F_1}} : F_1 \subsetneq F \text{ a subfield}\}.$$

Note that if F has no proper subfields, the only possible F_1 is the field $\mathbb{Q}$, and in this case $\kappa' = 1$. Also note that by Theorem 1.4, $\kappa' > \kappa(\Lambda)$.

We now claim that for any sequence $\Lambda_k \to \Lambda$, such that $\Lambda_k \notin A\Lambda$, we have $\liminf_k \kappa(\Lambda_k) \geq \kappa'$. Take a subsequence along which $\kappa(\Lambda_k)$ converges. Applying Theorem 7.1 in the special case $H = A$, after passing to a further subsequence we find that there is a subfield $F_1 \subsetneq F$ such that any lattice in $H'\Lambda$ is an accumulation point of a sequence of the form $a_k\Lambda_k$. Here H' is the equiblock group corresponding to F_1 under Corollary 5.1. In particular, we can choose $\Lambda_{\max}$, a lattice realizing the maximal value of κ on the homogeneous subset $H'\Lambda$ as in Theorem 3.1, as a limit point of $a_k\Lambda_k$. In view of Proposition 2.1,

$$\kappa' \leq \kappa(\Lambda_{\max}) \leq \lim \kappa(a_k\Lambda_k) = \lim \kappa(\Lambda_k).$$

Now to prove local isolation, note that Definition 1.8 is satisfied with $\varepsilon_0 \stackrel{\text{def}}{=} \kappa' - \kappa(\Lambda)$. This also implies strong isolation when F has no proper subfields. It remains to show that Λ is not strongly isolated when F does have a proper subfield F'. Indeed, in this case $\kappa' < 1$ by Theorem 1.4. Letting H' denote the block group corresponding to F', we find from Theorem 3.1 that there is a dense collection of lattices $\Lambda' \in H'\Lambda$, for which $\kappa(\Lambda') = \kappa'$. This means that Λ is not strongly isolated. $\square$

Proof of Corollary 1.11. We first recall that for any n there is a totally real number field F of degree n without proper subfields. Indeed, by [KM, Prop. 2], for any n there is a totally real Galois extension K of $\mathbb{Q}$ with Galois group $\mathcal{S}_n$ (the full permutation group of $\{1, \ldots, n\}$). The subgroup $G_0 \cong \mathcal{S}_{n-1}$ fixing the element 1 is a maximal subgroup of index n, so by the Galois correspondence, the subfield F of K fixed by G_0 has the required properties.

Now let Λ be a number field lattice in $\mathbb{R}^n$ arising from such a field F via the construction as in (14) of §5.2. By Theorem 1.9, Λ is strongly isolated, and by Proposition 5.6 (applied to $H_1 = A, H_2 = G$), the set of number field lattices with associated field F is dense in $\mathcal{L}_n$. $\square$

We now give a variant of Definition 1.8. Let $\Lambda \in \mathcal{L}_n$ and let $H \subset G$ be a subgroup containing A. Given $\varepsilon_0 > 0$, we say that Λ is ε_0-*isolated relative to* H if for any $0 < \varepsilon < \varepsilon_0$ there is a neighborhood $\mathcal{U}$ of Λ in $\mathcal{L}_n$, so that for any $\Lambda' \in \mathcal{U} \smallsetminus H\Lambda$, $\kappa(\Lambda') > \kappa(\Lambda) + \varepsilon$. We will say that Λ

is *locally isolated relative to* H if it is ε_0-isolated relative to H for some $\varepsilon_0 > 0$.

With this definition we prove the following result, of which Theorem 1.13 is a special case:

Theorem 7.2. *Let* μ *be a homogeneous* A*-invariant probability measure which corresponds to the homogeneous space* $H\Lambda_0$ *with* $A \subsetneq H \subsetneq G$*. Then for any* $\Lambda \in H\Lambda_0$ *for which* $\overline{A\Lambda} = H\Lambda_0$ *(in particular, for* μ *-almost any* Λ*) the following assertions hold:*

1. Λ *is not locally isolated.*
2. Λ *is* ε_0*-locally isolated relative to* H*, for*

$$\varepsilon_0 \overset{\mathrm{def}}{=} \min\{\kappa_\nu : {}^{\nu \text{ is a homogeneous } A\text{-invariant probability}}_{\text{measure with } \mathrm{supp}(\mu)\subsetneq \mathrm{supp}(\nu)}\} - \kappa(\Lambda).$$

Since there are only finitely many equiblock groups that contain H, there are only finitely many measures ν that can appear in the minimum defining ε_0 above. By Theorem 1.4 we see that indeed $\varepsilon_0 > 0$.

Proof. Since $A \subsetneq H$, (1) is immediate from Theorem 3.1, taking a sequence of generic elements in $H\Lambda \smallsetminus A\Lambda$ tending to Λ. The proof of assertion (2) is identical to the proof of Theorem 1.9, except that in applying Theorem 7.1, we use H in place of A. □

Proof of Theorem 1.14. By Proposition 5.9 there are number field lattices in $H\Lambda$, and by Proposition 5.6 the collection of number field lattices in $H\Lambda$ is dense. Let $\Lambda_0 \in H\Lambda$ be a lattice realizing the generic value κ_μ and choose $\Lambda_k \to \Lambda_0$ a sequence of number field lattices from within $H\Lambda$. On the one hand, by Theorems 3.1 and 1.4 we know that $\kappa(\Lambda_k) < \kappa_\mu$. On the other hand, by Proposition 2.1, $\liminf_k \kappa(\Lambda_k) \geq \kappa_\mu$. It follows that the sequence $\kappa(\Lambda_k)$ converges to κ_μ and after possibly taking a subsequence, we may assume that $\kappa(\Lambda_k) \nearrow \kappa_\mu$. Finally, by Proposition 5.10, these values belong to the reduced Mordell-Gruber spectrum. □

Proof of Theorem 1.16. Given t, let $\mathbb{Q} \subsetneq F_1 \subsetneq \cdots \subsetneq F_t$ be a tower of totally real fields, and let $n = \deg(F_t/\mathbb{Q})$. Let $\Lambda \subset \mathcal{L}_n$ be a number field lattice corresponding to F_t, constructed via (14) for some rank n subgroup $L \subset F_t$. Then each of the F_i is obtained as $\mathcal{A}_\Lambda(\mathcal{P}_i)$ for some partition $\mathcal{P}_i$, and by Corollary 5.1, the corresponding groups $H_i \overset{\mathrm{def}}{=} H(\mathcal{P}_i)$ satisfy $A = H_t \subsetneq H_{t-1} \subsetneq \cdots \subsetneq H_1 \subsetneq G$. For each i, let $H_i\Lambda$ be the corresponding finite-volume homogeneous subspace. Denote $\mathrm{SL}(V_\Lambda) \overset{\mathrm{def}}{=} G \cap \mathrm{GL}(V_\Lambda)$, i.e., the set of elements of G which are rational with respect to the $\mathbb{Q}$-structure induced by Λ. For each $q \in \mathrm{SL}(V_\Lambda)$ and each i, the orbit $H_iq\Lambda$ is also a homogeneous subspace, since q commensurates $\mathrm{Stab}_G(\Lambda)$. Let $\kappa_i(q)$ denote the generic value of κ, as in Theorem 3.1, on the homogeneous subspace $H_i q\Lambda$. We will show by induction that each $\kappa_{t-i}(q)$ belongs to $\widehat{\mathbf{MG}}_n^{(i)}$.

Suppose first that $i = 1$. Then each $H_{t-1}q\Lambda$ is a homogeneous subspace, which contains the compact A-orbits $Aq'\Lambda$, for all $q' \in \mathrm{SL}(V_\Lambda) \cap H_{t-1}q$. Since $\mathrm{SL}(V_\Lambda)$ is a group and $\mathrm{SL}(V_\Lambda) \cap H_i$ is dense in each H_i, the set of such q' is dense in each $H_{t-1}q$. Therefore, repeating the argument proving Theorem 1.14, we find that each $\kappa_{t-1}(q)$ is a limit of an increasing sequence from $\widehat{\mathbf{MG}}_n^{(i)}$. For the case of general i we argue in the same way, taking all $q' \in \mathrm{SL}(V_\Lambda) \cap H_{t-i+1}q$, and using the values of κ corresponding to $H_{t-i}q'\Lambda$ to approximate the value $H_{t-i+1}q\Lambda$. $\square$

8. The case $n = 2$

For a lattice $\Lambda \subset \mathbb{R}^n$, we denote

$$\lambda(\Lambda) \stackrel{\mathrm{def}}{=} \inf \left\{ \left| \prod x_i \right| : (x_1, \ldots, x_n) \in \Lambda \smallsetminus \{0\} \right\}.$$

The following was proved in [G]:

Proposition 8.1 (Gruber)**.** *For a lattice Λ of dimension 2, $\kappa(\Lambda) < 1 \iff \lambda(\Lambda) > 0$.*

Remark 8.2. Using the results of the previous sections, it is not hard to show that Gruber's result is not valid for general n. Indeed, Mahler's compactness criterion (Proposition 2.2) implies that the condition $\lambda(\Lambda) > 0$ is equivalent to the boundedness of the A-orbit of Λ in $\mathcal{L}_n$. Now let n be composite and let μ be a homogeneous measure on $\mathcal{L}_n$ supported on intermediate lattices which are not number field lattices. In light of [LW, Step 6.3] and Proposition 3.6, for almost any $\Lambda \in \mathrm{supp}(\mu)$, $A\Lambda$ is not bounded, so that $\lambda(\Lambda) = 0$. However by Theorem 1.4, $\kappa(\Lambda) < 1$.

Proof of Theorem 1.12. Let Λ be a lattice in dimension 2, with $\kappa(\Lambda) < 1$. We wish to show that it is not strongly isolated. In view of Proposition 8.1, we know that $\lambda(\Lambda) > 0$, and it suffices to show that there is a bounded A-orbit $A\Lambda_0$ which contains $A\Lambda$ in its closure but is not equal to $A\Lambda$. Since $n = 2$, the A-action is the geodesic flow on the unit tangent bundle to the modular surface, and the existence of such orbits is well-known using symbolic dynamics. More specifically, using the viewpoint of [AF], for any lattice $\Lambda \in \mathcal{L}_2$, let α, ω be two real numbers which are endpoints of the infinite geodesic through a lift of the tangent vector corresponding to Λ in the upper half-plane. Since $A\Lambda$ is bounded, the continued fractions coefficients of the numbers α, ω are bounded, say by a number k. Denote these coefficients by $\alpha = [a_{-1}, a_{-2}, \ldots]$ and $\omega = [a_0, a_1, a_2, \ldots]$. Now let $\alpha' \stackrel{\mathrm{def}}{=} [k+1, k+1, \ldots]$ and

$$\omega' \stackrel{\mathrm{def}}{=} [a_0, a_{-1}, a_0, a_1, a_{-2}, a_{-1}, a_0, a_1, a_2, a_{-3}, a_{-2}, \ldots].$$

That is, the bi-infinite word obtained by concatenating the expansions of α and ω is in the orbit-closure, under the shift, of the the bi-infinite word obtained by concatenating the expansions of α', ω'.

In view of the symbolic coding of the geodesic flow [AF], the closure of the projection in $\mathcal{L}_2$ of the geodesic with endpoints α', ω' contains the projection of the geodesic with endpoints α, ω. Since the digits of α' are greater than k, the two orbits are distinct. Since all digits of α', ω' are bounded by $k+1$, the corresponding A-orbit has $\kappa < 1$. $\square$

Theorem 1.14 concerns the existence of accumulation points for $\widehat{\mathbf{MG}}_n$ besides 1, for $n \geq 3$. As we now explain, Proposition 8.1 can be used to settle this question in dimension 2.

Proposition 8.3. *The set* $\mathbf{MG}_2 = \widehat{\mathbf{MG}}_2$ *has accumulation points smaller than 1.*

Proof. Let $\mathbf{Bad}_k$ denote the set of real numbers x whose continued fraction coefficients $a_1(x), a_2(x), \ldots$ are bounded above by k. Then it is well-known that $\cup_{k\geq 1}\mathbf{Bad}_k$ contains all real quadratic irrationals. Given a real quadratic irrational x, let $L = \mathbb{Z} \oplus \mathbb{Z}x$ be an additive subgroup in the corresponding quadratic field $\mathbb{Q}(x)$, and $\Lambda = \Lambda(x) \in \mathcal{L}_2$ be the lattice in dimension 2, constructed via (14). Then, as is well-known (and is a very special case of Corollary 4.10) the orbit $A\Lambda(x)$ is compact. The inequalities of [G] imply that a uniform bound on the continued fraction coefficients of x imply a uniform bound on the Mordell constant; in particular, for any k there is $\kappa_0 < 1$ so that if $x \in \mathrm{Bad}_k$ is a quadratic irrational, then $\kappa(\Lambda(x)) \leq \kappa_0$.

It is known that there is k such that $\mathbf{Bad}_k$ contains a sequence (x_n) of quadratic irrationals, for which the fields $F_n \stackrel{\mathrm{def}}{=} \mathbb{Q}(x_n)$ are distinct quadratic fields. Indeed, as explained to the authors by Dmitry Kleinbock, one can take $k = 2$. By [CF, Theorem 1.6], the quadratics in $\mathbf{Bad}_2$ are the numbers $\sqrt{(3m-2)(3m+2)}/m$, and it follows from Dirichlet's theorem on primes in arithmetic progressions, that among these, numbers belonging to infinitely many distinct fields arise.

For each n, as in Step 1 of Theorem 6.8, there are $v_1^{(n)}, v_2^{(n)}$ which are locking points for $\Lambda(x_n)$. Let $\sigma_1^{(n)}, \sigma_2^{(n)}$ be the two field embeddings of F_n. After applying an element of A, we find by (14) that there are α_n, β_n in F_n such that $v_1^{(n)} = \left(\sigma_1^{(n)}(\alpha_n), \sigma_2^{(n)}(\alpha_n)\right)$ and $v_2^{(n)} =$

$\left(\sigma_1^{(n)}(\beta_n), \sigma_2^{(n)}(\beta_n)\right)$, so that

$$\kappa_n \stackrel{\text{def}}{=} \kappa(\Lambda(x_n)) = \sigma_1^{(n)}(\alpha_n) \cdot \sigma_2^{(n)}(\beta_n).$$

Since α_n, β_n span F_n, they are linearly independent over $\mathbb{Q}$. On the other hand $\sigma_1^{(n)}(\alpha_n)\sigma_2^{(n)}(\alpha_n) \in \mathbb{Q}$. This implies that κ_n is irrational. Since the κ_n belong to distinct quadratic fields, they are therefore distinct. So the sequence (κ_n) is an infinite sequence in $\mathbf{MG}_2$, bounded above by $\kappa_0 < 1$. This implies that $\mathbf{MG}_2$ has a limit point smaller than 1. □

Remark 8.4. 1. By a similar argument, in order to show that there are infinitely many distinct accumulation points in $\mathbf{MG}_2$, it suffices to construct infinitely many disjoint finite blocks of natural numbers B_n, and for each n, an infinite sequence of quadratics in distinct fields, whose continued fractions coefficients lie in B_n.

2. Nikolay Moshchevitin has directed our attention to the work of B. Diviš [D]. Diviš studied the so-called 'Dirichlet spectrum', namely he defined

$$d(x) = \sup_{t \geq 1} \min_{p \in \mathbb{Z},\, q \in \mathbb{N}, q \leq t} t\,|qx - p|, \quad \text{and } \mathbb{D} \stackrel{\text{def}}{=} \{d(x) : x \in \mathbb{R}\},$$

and showed that $\mathbb{D}$ contains an interval and is not closed. The Dirichlet spectrum can be interpreted in terms of the one-sided geodesic trajectory $\{g_t\Lambda_x : t \geq 0\}$, where Λ_x is the lattice spanned by $(1, 0)$ and $(x, 1)$, while the Mordell-Gruber spectrum can be interpreted in terms of the full trajectory $\{g_t\Lambda : t \in \mathbb{R}\}$ of a lattice Λ. We suspect that the arguments of Diviš can be adapted to show that $\mathbf{MG}_2$ contains an interval and is not closed.

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