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Number theory/Geometry

A volume estimate for the set of stable lattices

Une estimation du volume de l'ensemble des réseaux stables

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ABSTRACT

We show that in high dimensions the set of stable lattices is almost of full measure in the space of unimodular lattices.

RÉSUMÉ

Nous montrons qu'en grande dimension, l'ensemble des réseaux stables est de mesure presque pleine dans l'espace des réseaux unimodulaires.

Let $G \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{R})$, $\Gamma \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{Z})$, and let $A \subset G$ denote the subgroup of diagonal matrices with positive entries. The quotient space $\mathcal{L}_n \stackrel{\text{def}}{=} G/\Gamma$ is naturally identified with the space of unimodular lattices in $\mathbb{R}^n$, and the group G (and any of it subgroups) acts via left translations, or equivalently, by acting on lattices via its linear action on $\mathbb{R}^n$. A lattice Λ is called *stable* if for any subgroup $\Delta \subset \Lambda$, one has $\mathrm{vol}(\Delta \otimes \mathbb{R}/\Delta) \geq 1$ (in the literature the term *semi-stable* is also used), and we denote the set of stable lattices by $\mathcal{S}^{(n)}$.

A central problem is to understand the orbits of the A-action on $\mathcal{L}_n$. In [1] we proved that for any lattice $\Lambda \in \mathcal{L}_n$, the orbit-closure $\overline{A\Lambda}$ contains a stable lattice. This result reduces the proof of Minkowski's conjecture on the product of inhomogeneous linear forms to that of estimating the Euclidean covering radius of stable lattices (see [1] for details). Understanding stable lattices is therefore a natural problem due to its connection both with well-studied problems in the geometry of numbers, and with dynamics of the A-action. Although $\mathcal{S}^{(n)}$ is compact (while $\mathcal{L}_n$ is not), in this note we show that $\mathcal{S}^{(n)}$ has almost full measure with respect to the natural probability measure on $\mathcal{L}_n$, for large n. Moreover the convergence to full measure is very fast. This answers a question we were asked by G. Harder, and can be viewed as a manifestation of the concentration of mass along the equator in high dimensional Euclidean balls.

We will prove the following.

Theorem 1. *Let m denote the G-invariant probability measure on $\mathcal{L}_n$ derived from Haar measure on G, and let $\mathcal{S}^{(n)}$ denote the subset of stable lattices in $\mathcal{L}_n$. Then there is a constant $C > 0$ such that for all sufficiently large n,*

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$$m\big(\mathcal{L}_n \smallsetminus \mathcal{S}^{(n)}\big) \le \left(\frac{C}{n}\right)^{\frac{n-1}{2}}.$$

In particular $m(\mathcal{S}^{(n)}) \longrightarrow 1$ *as* $n \to \infty$.

For $\Lambda \in \mathcal{L}_n$ and a subgroup $\Delta \subset \Lambda$, we denote by $r(\Delta)$ its *rank* and by $|\Delta|$ its *covolume* in the Euclidean subspace $\Delta \otimes \mathbb{R} \subset \mathbb{R}^n$. For $k = 1, \dots, n-1$ let us denote $\mathcal{V}_k(\Lambda) \stackrel{\text{def}}{=} \{|\Delta|^{1/k} : \Delta \subset \Lambda, r(\Delta) = k\}$ and $\alpha_k(\Lambda) = \min \mathcal{V}_k(\Lambda)$ so that Λ is stable if and only if $\alpha_k(\Lambda) \ge 1$ for $k = 1, \dots, n-1$. Let

$$\mathcal{S}_k^{(n)}(t) \stackrel{\text{def}}{=} \big\{x \in \mathcal{L}_n : \alpha_k(x) \ge t\big\}, \qquad \mathcal{S}_k^{(n)} \stackrel{\text{def}}{=} \mathcal{S}_k^{(n)}(1).$$

With this notation $\mathcal{S}^{(n)} = \bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}$. We will show:

Proposition 2. *There is* $C > 0$ *such that for all sufficiently large* n*, and all* $k \in \{1, \dots, n-1\}$*,*

$$m\big(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}\big) \le \frac{1}{n}\left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}. \tag{1}$$

Proof of Theorem 1. For $n > C$, the largest value of $(\frac{C}{n})^{\frac{k(n-k)}{2}}$ is attained when $k = 1$ and $k = n-1$. Therefore (1) implies

$$\begin{aligned} m\big(\mathcal{L}_n \smallsetminus \mathcal{S}^{(n)}\big) &= m\left(\mathcal{L}_n \smallsetminus \bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}\right) = m\left(\bigcup_{k=1}^{n-1} \mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}\right) \\ &\le \frac{n-2}{n}\left(\frac{C}{n}\right)^{\frac{n-1}{2}} \le \left(\frac{C}{n}\right)^{\frac{n-1}{2}}. \qquad \square \end{aligned}$$

We will also show:

Proposition 3. *There is* $C_1 > 0$ *such that if we set*

$$t_k = t(n,k) \stackrel{\text{def}}{=} \left(\frac{n}{C_1}\right)^{\frac{n-k}{2n}}, \tag{2}$$

then

$$\max_{k=1,\dots,n-1} m\big(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t_k)\big) = o\left(\frac{1}{n}\right).$$

In particular, $m(\bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}(t_k)) \to_{n\to\infty} 1$.

Remarks. 1. Let us define $\bar{\alpha}_{n,k} \stackrel{\text{def}}{=} \sup\{\alpha_k(\Lambda) : \Lambda \in \mathcal{L}_n\}$. These quantities are powers of the so-called *Rankin constants* or *generalized Hermite constants*, usually denoted by $\gamma_{n,k}$ (see [5]), namely they are related by

$$\bar{\alpha}_{n,k}^{2k} = \gamma_{n,k}. \tag{3}$$

The origin of this exponent $2k$ is the $1/k$ in the definition of $\mathcal{V}_k$, which we have imposed so that the functions α_k behave nicely with respect to homothety. This normalization has the additional advantage that the growth rate of the different $\bar{\alpha}_{n,k}$ (as a function of n) becomes the same for all k. Namely [5, Cor. 2] and (3) show that $\log \bar{\alpha}_{n,k} = \frac{1}{2}\log n + O(1)$ (where the implicit constant depends on k).

2. It seems plausible that most lattices come close to realizing the Rankin constants, that is, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} m\big(\big\{\Lambda \in \mathcal{L}_n : \forall k,\ \alpha_k(\Lambda) > \bar{\alpha}_{n,k} - \varepsilon\big\}\big) = 1.$$

Combined with the result of Thunder mentioned above, Proposition 3 may be viewed as supporting evidence for such a conjecture.

3. Andreas Strömbergsson has informed us that he has proved (relying on [3]) that for k fixed, with probability tending to 1 as $n \to \infty$ (with respect to the measure m on $\mathcal{L}_n$), a lattice Λ has $\alpha_k(\Lambda) = \sqrt{\frac{n}{2\pi e}} + o(1)$.

4. We take this opportunity to formulate an analogous question regarding the *covering radius*; namely, is it true that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} m\left\{\Lambda \in \mathcal{L}_n : \operatorname{covrad}(\Lambda) < \inf_{\Lambda' \in \mathcal{L}_n} \operatorname{covrad}\left(\Lambda'\right) + \varepsilon\right\} = 1,$$

where

$$\operatorname{covrad}(\Lambda) = \inf\left\{r > 0 : \mathbb{R}^n = \Lambda + B(0,r)\right\}$$

and $B(0,r)$ is the Euclidean ball of radius r around the origin. A similar question regarding a different notion of covering radius was posed in [4].

The proof of Propositions 2 and 3 relies on Thunder's work and on a variant of Siegel's formula [2] that relates the Lebesgue measure on $\mathbb{R}^n$ and the measure m on $\mathcal{L}_n$. We now review Siegel's method and Thunder's results.

In the sequel, we consider $n \geq 2$ and $k \in \{1, \dots, n-1\}$ as fixed and omit, unless there is risk of confusion, the symbols n and k from the notation. Consider the (set valued) map $\Phi = \Phi_k^{(n)}$ that assigns to each lattice $\Lambda \in \mathcal{L}_n$ the following subset of the exterior power of $\bigwedge^k \mathbb{R}^n$:

$$\Phi(\Lambda) \stackrel{\text{def}}{=} \left\{\pm w_\Delta : \Delta \subset \Lambda \text{ a primitive subgroup with } r(\Delta) = k\right\},$$

where $w_\Delta \stackrel{\text{def}}{=} v_1 \wedge \cdots \wedge v_k$ and $\{v_i\}_{i=1}^k$ is a basis for Δ (note that w_Δ is well-defined up to sign, and $\Phi(\Lambda)$ contains both possible choices). Let

$$\mathcal{V} = \mathcal{V}_k^{(n)} \stackrel{\text{def}}{=} \left\{v_1 \wedge \cdots \wedge v_k : v_i \in \mathbb{R}^n\right\} \smallsetminus \{0\}$$

be the variety of pure tensors in $\bigwedge \mathbb{R}^n$. For any compactly supported bounded Riemann integrable[1] function f on $\mathcal{V}$, set

$$\hat{f} : \mathcal{L}_n \to \mathbb{R}, \qquad \hat{f}(\Lambda) \stackrel{\text{def}}{=} \sum_{w \in \Phi(\Lambda)} f(w). \tag{4}$$

Then it is known (see [6, Lemma 2.4.2]) that the (finite) sum (4) defines a function in $L^1(\mathcal{L}_n, m)$. This allows us to define a Radon measure $\theta = \theta_k^{(n)}$ on $\mathcal{V}$ by the formula:

$$\int_{\mathcal{V}} f \,\mathrm{d}\theta \stackrel{\text{def}}{=} \int_{\mathcal{L}_n} \hat{f} \,\mathrm{d}m, \quad \text{for } f \in C_c(\mathcal{V}). \tag{5}$$

Write $G = G_n \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{R})$. There is a natural transitive action of G_n on $\mathcal{V}$ and the stabilizer of $e_1 \wedge \cdots \wedge e_k$ is the subgroup

$$H = H_k^{(n)} \stackrel{\text{def}}{=} \left\{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in G : A \in G_k, D \in G_{n-k}\right\}.$$

We therefore obtain an identification $\mathcal{V} \simeq G/H$ and view θ as a measure on G/H. It is well known (see, e.g., [6]) that up to a proportionality constant, there exists a unique G-invariant measure $m_{G/H}$ on G/H; moreover, given Haar measures m_G, m_H on G and H, respectively, there is a unique normalization of $m_{G/H}$ such that for any $f \in L^1(G, m_G)$

$$\int_G f \,\mathrm{d}m_G = \int_{G/H} \int_H f(gh) \mathrm{d}m_H(h) \mathrm{d}m_{G/H}(gH). \tag{6}$$

We choose the Haar measure m_G so that it descends to our probability measure m on $\mathcal{L}_n$; similarly, we choose the Haar measure m_H so that the periodic orbit $H\mathbb{Z}^n \subset \mathcal{L}_n$ has volume 1. These choices of Haar measures determine our measure $m_{G/H}$ unequivocally. It is clear from the defining formula (5) that θ is G-invariant and therefore the two measures $m_{G/H}, \theta$ are proportional. In fact (see [2] for the case $k = 1$ and [6, Lemma 2.4.2] for the general case),

$$m_{G/H} = \theta. \tag{7}$$

For $t > 0$, let $\chi = \chi_t : \mathcal{V} \to \mathbb{R}$ be the restriction to $\mathcal{V}$ of the characteristic function of the ball of radius t around the origin, in $\bigwedge^k \mathbb{R}^n$, with respect to the natural inner product obtained from the Euclidean inner product on $\mathbb{R}^n$. Note that $\hat{\chi}(x) = 0$ if and only if $x \in \mathcal{S}_k^{(n)}(t^{1/k})$ and furthermore, $\hat{\chi}(x) \geq 1$ if $x \in \mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t^{1/k})$. It follows that

$$m\left(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t)\right) \leq \int_{\mathcal{L}_n} \widehat{(\chi_{t^k})} \mathrm{d}m = \int_{\mathcal{V}} \chi_{t^k} \mathrm{d}\theta. \tag{8}$$

[1] I.e. the measure of points at which f is not continuous is zero.

Let V_j denote the volume of the Euclidean unit ball in $\mathbb{R}^j$ and let ζ denote the Riemann zeta function. We will use an unconventional convention $\zeta(1) = 1$, which will make our formulae simpler. For $j \geq 1$, define

$$R(j) \stackrel{\text{def}}{=} \frac{jV_j}{\zeta(j)} \quad \text{and} \quad B(n,k) \stackrel{\text{def}}{=} \frac{\prod_{j=1}^{n} R(j)}{\prod_{j=1}^{k} R(j) \prod_{j=1}^{n-k} R(j)}.$$

The following is [5, Lemma 5]:

Theorem 4 *(Thunder). For $t > 0$, we have $\int_{\mathcal{V}} \chi_t \, \mathrm{d}m_{G/H} = B(n,k)\frac{t^n}{n}$.*

(Note that in Thunder's notation, by [5, §4], $c(n,k) = B(n,k)/n$.)

We will need to bound $B(n,k)$.

Lemma 5. *There is $C > 0$ so that for all large enough n and all $k = 1, \ldots, n-1$,*

$$B(n,k) \leq \left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}. \tag{9}$$

Proof. In this proof $c_0, c_1, \ldots$ are constants independent of n, k, j. Because of the symmetry $B(n,k) = B(n, n-k)$, it is enough to prove (9) with $k \leq \frac{n}{2}$. Using the formula $V_j = \frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}$, we obtain:

$$\begin{aligned} B(n,k) &= \prod_{j=1}^{k} \frac{R(n-k+j)}{R(j)} = \prod_{j=1}^{k} \frac{\zeta(j)(n-k+j)\frac{\pi^{(n-k+j)/2}}{\Gamma(\frac{n-k+j}{2}+1)}}{\zeta(n-k+j)j\frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}} \\ &= \prod_{j=1}^{k} \frac{\zeta(j)}{\zeta(n-k+j)} \cdot \frac{n-k+j}{j} \cdot \pi^{\frac{n-k}{2}} \cdot \frac{\Gamma(\frac{j}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)}. \end{aligned}$$

Note that $\zeta(s) \geq 1$ is a decreasing function of $s > 1$, so (recalling our convention $\zeta(1) = 1$) $\frac{\zeta(j)}{\zeta(n-k+j)} \leq c_0 \stackrel{\text{def}}{=} \zeta(2)$. It follows that for all large enough n and for any $1 \leq j \leq k$,

$$\frac{\zeta(j)}{\zeta(n-k+j)} \cdot \frac{n-k+j}{j} \cdot \pi^{\frac{n-k}{2}} \leq c_0 n \pi^{\frac{n-k}{2}} \leq 4^{\frac{n-k}{2}}. \tag{10}$$

According to Stirling's formula, there are positive constants c_1, c_2 such that for all $x \geq 2$,

$$c_1\sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x \leq \Gamma(x) \leq c_2\sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x.$$

We set $u \stackrel{\text{def}}{=} \frac{j}{2} + 1$ and $v \stackrel{\text{def}}{=} \frac{n-k}{2}$, so that $u + v \geq \frac{n-1}{4}$, and obtain

$$\begin{aligned} \frac{\Gamma(\frac{j}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)} &= \frac{\Gamma(u)}{\Gamma(u+v)} \leq \frac{c_2}{c_1}\sqrt{\frac{u+v}{u}} \frac{u^u}{(u+v)^{u+v}} \frac{e^{u+v}}{e^u} \\ &\leq c_3 e^v \frac{u^{u-1/2}}{(u+v)^{u+v-1/2}} = c_3\left(\frac{e}{u+v}\right)^v \frac{1}{(1+\frac{v}{u})^{u-1/2}} \\ &\leq c_3\left(\frac{4e}{n-1}\right)^{\frac{n-k}{2}}. \end{aligned} \tag{11}$$

Using (10) and (11) we obtain

$$B(n,k) \leq \left[c_3 4^{\frac{n-k}{2}}\left(\frac{4e}{n-1}\right)^{\frac{n-k}{2}}\right]^k = \left[c_3\left(\frac{16e}{n-1}\right)^{\frac{n-k}{2}}\right]^k.$$

So taking $C > 16c_3 e$ we obtain (9) for all large enough n. $\square$

Proof of Propositions 2 and 3. Let C be as in Lemma 5 and let $C_1 > C$. For Proposition 3, using (8), (7) and Theorem 4, for all sufficiently large n, we have:

$$m\big(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t_k)\big) \le B(n,k)\frac{t_k^{kn}}{n}$$
$$\le \frac{1}{n}\left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}\left(\frac{n}{C_1}\right)^{\frac{k(n-k)}{2}} = \frac{1}{n}\left(\frac{C}{C_1}\right)^{\frac{k(n-k)}{2}}.$$

Multiplying by n and taking the maximum over k, we obtain:

$$n \max_{k=1,\dots,n} m\big(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t_k)\big) \le \left(\frac{C}{C_1}\right)^{\frac{n-1}{2}} \to_{n\to\infty} 0.$$

The proof of Proposition 2 is identical using $t = 1$ instead of t_k. □

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