DYNAMICS ON THE SPACE OF 2-LATTICES IN 3-SPACE

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Dedicated to Ponyo

Abstract. We study the dynamics of $\mathrm{SL}_3(\mathbb{R})$ and its subgroups on the homogeneous space X consisting of homothety classes of rank-2 discrete subgroups of $\mathbb{R}^3$. We focus on the case where the acting group is Zariski dense in either $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SO}(2,1)(\mathbb{R})$. Using techniques of Benoist and Quint we prove that for a compactly supported probability measure μ on $\mathrm{SL}_3(\mathbb{R})$ whose support generates a group which is Zariski dense in $\mathrm{SL}_3(\mathbb{R})$, there exists a unique μ-stationary probability measure on X. When the Zariski closure is $\mathrm{SO}(2,1)(\mathbb{R})$ we establish a certain dichotomy regarding stationary measures and discover a surprising phenomenon: The Poisson boundary can be embedded in X. The embedding is of algebraic nature and raises many natural open problems. Furthermore, motivating applications to questions in the geometry of numbers are discussed.

1. Introduction

1.1. **A motivating conjecture.** We begin by stating the conjecture which motivated this paper and remains unsolved. Let X_2 be the space of lattices in $\mathbb{R}^2$ identified up to scaling. The quotient $\mathrm{O}_2(\mathbb{R})\backslash X_2$ of X_2 by the action of the orthogonal group is thought of as the space of *shapes of 2-lattices*. Given a rank-2 discrete subgroup $\Lambda \subset \mathbb{R}^3$ – hereafter known as a 2-lattice – we define its shape $\mathbf{s}(\Lambda)$ to be the point of $\mathrm{O}_2(\mathbb{R})\backslash X_2$ corresponding to an image of Λ in X_2 obtained by choosing an arbitrary isometry between the plane spanned by Λ and $\mathbb{R}^2$.

Conjecture 1.1. *Consider the signature (2,1) quadratic form $Q(v_1, v_2, v_3) := 2v_1v_3 - v_2^2$ and the variety $V_Q^1 := \{v \in \mathbb{R}^3 : Q(v) = 1\}$. Let $V_Q^1(\mathbb{Z}) := V_Q^1 \cap \mathbb{Z}^3$ denote the collection of integer points on V_Q^1. Then, the collection of* orthogonal shapes

$$\{\mathbf{s}(\mathbb{Z}^3 \cap v^\perp) : v \in V_Q^1(\mathbb{Z})\}$$

is dense in $\mathrm{O}_2(\mathbb{R})\backslash X_2$.

Currently it is not even known that the above set is unbounded. Conjecture 1.1 was motivated by a conjecture of Furstenberg which is related to a conjecture about cubic irrationals discussed in Appendix A (Conjecture A.3). Using duality it is easy to see that Conjecture 1.1 would follow from the density of the collection $\{\mathbf{s}(g\Lambda_{v_1}) : g \in \mathrm{SO}(Q)(\mathbb{Z})\}$, where $\Lambda_{v_1} = \mathbb{Z}^3 \cap v_1^\perp$ for $v_1 = (1, 1, 1) \in V_Q^1(\mathbb{Z})$. See Figure 1 for compelling evidence towards Conjecture 1.1. In our figures we plot some numerical experiments. Since the more familiar space $\mathrm{PSO}_2(\mathbb{R})\backslash X_2$ is a double cover of $\mathrm{O}_2 \backslash X_2$, we lift the plots to this space.

Motivated by the above discussion, we can now present a corollary of one of our main results. We consider the case where $\mathrm{SO}(Q)(\mathbb{Z})$ is replaced by a Zariski dense subgroup of $\mathrm{SL}_3(\mathbb{R})$.

Theorem 1.2. *Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be a compactly generated Zariski dense subgroup and let $\Lambda < \mathbb{R}^3$ be a rank-2 discrete subgroup. Then the collection of shapes $\{\mathbf{s}(g\Lambda) : g \in \Gamma\}$ is dense in $\mathrm{O}_2(\mathbb{R})\backslash X_2$.*

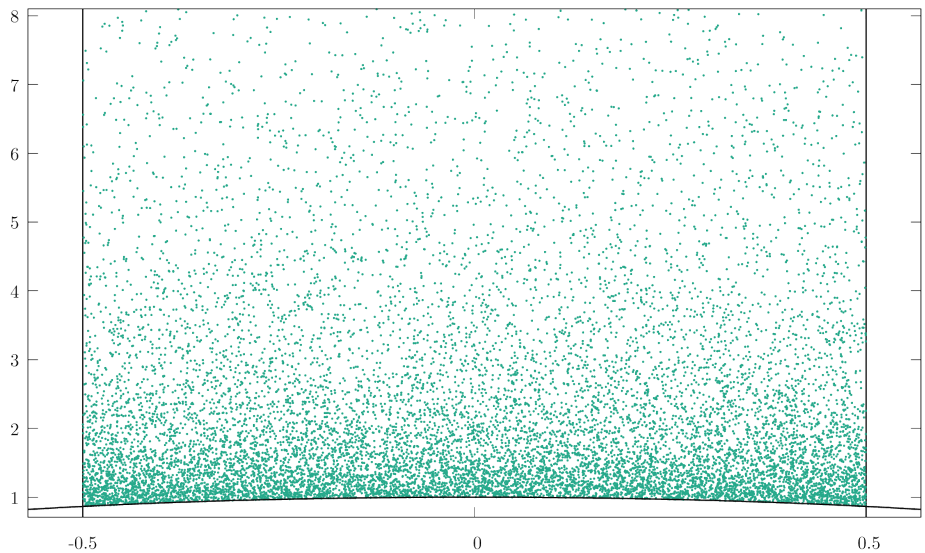

FIGURE 1. Plot of $\approx 15,000$ points in $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H} \simeq \mathrm{PSO}_2(\mathbb{R})\backslash X_2$ corresponding to the shapes $\mathbf{s}(g\Lambda_{v_1})$ where $g \in \mathrm{SO}(Q)(\mathbb{Z})$ is chosen 'randomly'.

Remark 1.3. *A much stronger statement holds in the setting of Theorem 1.2. Let μ be a compactly supported probability measure on $\mathrm{SL}_3(\mathbb{R})$ such that the group generated by the support of μ is Γ. Then for $\mu^{\otimes\mathbb{N}}$-almost every $(g_1, g_2, \dots) \in \mathrm{SL}_3(\mathbb{R})^{\mathbb{N}}$ the sequence $\mathbf{s}(g_n \cdots g_1\Lambda)$ is equidistributed in $\mathrm{O}_2(\mathbb{R})\backslash X_2$ with respect to the uniform measure on $\mathrm{O}_2(\mathbb{R})\backslash X_2$.*

Our attempt towards proving Conjecture 1.1 involves studying random walks on the space of 2-lattices. We build heavily on results and ideas from the seminal series of papers of Benoist and Quint [BQ11, BQ12, BQ13a, BQ13b] and prove two classification results regarding stationary measures on this space under assumptions on the acting group. Theorem 1.2 is an immediate consequence of Theorem 1.6 which is a strong classification theorem stating the uniqueness of a stationary probability measure – which we refer to as the *natural lift* – under the assumption that the acting measure generates a group which is Zariski dense in $\mathrm{SL}_3(\mathbb{R})$. The analogous classification for the case when the Zariski closure is $\mathrm{SO}(Q)(\mathbb{R})$ is weaker in the sense that sometimes there are stationary probability measures other than the natural lift. This is the reason we could not establish Conjecture 1.1, but it is not unlikely that further investigations of the structure of the space of ergodic μ-stationary probability measures will lead to the resolution of Conjecture 1.1. See Problem 1.13.

1.2. **Statements of results.** For a topological space Y we let $\mathcal{P}(Y)$ denote the space of Borel probability measures on Y. For $G \curvearrowright Y$ a continuous action of a topological group G and $\mu \in \mathcal{P}(G)$ we let $\mathcal{P}_\mu(Y)$ be the subset of $\mathcal{P}(Y)$ consisting of μ-stationary measures.

Henceforth we set

$$G := \mathrm{SL}_3(\mathbb{R})$$

and for $\mu \in \mathcal{P}(G)$

$$\Gamma_\mu := \langle \operatorname{supp} \mu \rangle$$

will be the group generated by the support of μ. A measure $\nu \in \mathcal{P}_\mu(X)$ is said to be μ-ergodic if the action of Γ_μ on (X, ν) is ergodic. It is a classical result of Furstenberg [Fur63b] (see [BQ16, Chapter 4] for a modern exposition) that if Γ_μ acts strongly irreducibly and proximally on $\mathbb{R}^3$, then $\mathcal{P}_\mu(\mathrm{Gr}_2(\mathbb{R}^3))$ consists of a single element. We will refer to it as the *Furstenberg measure* of μ on $\mathrm{Gr}_2(\mathbb{R}^3)$ and denote it by $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$.

Remark 1.4. *It will be important for us that the Furstenberg measure is atom free. This is ensured by the strong irreducibility assumption, since if there was an atom of $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ then the set of atoms with maximal weight is a finite Γ_μ-invariant set.*

We fix $\{e_1, e_2, e_3\}$ the standard orthonormal basis of unit vectors in $\mathbb{R}^3$. For $v \in \mathbb{R}^3$ and $1 \le i \le 3$ we will write $v_i := \langle v, e_i \rangle$. As before we consider the indefinite quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$Q(v) := 2v_1 v_3 - v_2^2. \tag{1.1}$$

Let H_μ denote the Zariski closure of Γ_μ. In what follows we will concentrate on two cases which will be referred to as Case I and Case II as follows:

$$H_\mu = \mathrm{SL}_3(\mathbb{R}) \tag{Case I}$$

$$H_\mu = \mathrm{SO}(Q)(\mathbb{R}). \tag{Case II}$$

In both of these cases it follows from a theorem of Gol'dsheid and Margulis (see [Abe08, Theorem 5.1] or [GM89]) that Γ_μ acts strongly irreducibly and proximally on $\mathbb{R}^3$. For the rest of the paper X will be the space of rank-2 discrete subgroups in $\mathbb{R}^3$ identified up to scaling. The linear G-action on $\mathbb{R}^3$ induces a transitive G-action on X endowing it with the structure of a homogeneous space. There is a natural projection

$$\pi : X \to \mathrm{Gr}_2(\mathbb{R}^3)$$

which sends an equivalence class of a 2-lattice to the plane it spans. We note that π is G-equivariant.

Given a rank-2 discrete subgroup $\Lambda \subset \mathbb{R}^3$ we denote its equivalence class modulo scaling by $[\Lambda]$. Abusing terminology we refer to both Λ and $[\Lambda]$ as a 2-lattice and to X as the space of 2-lattices in $\mathbb{R}^3$. For each plane $p \in \mathrm{Gr}_2(\mathbb{R}^3)$ the fibre $\pi^{-1}(p) \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. This identification is not canonical and depends on choosing a linear isomorphism between p and $\mathbb{R}^2$. Still, the unique $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ translates to a well defined probability measure $m_p \in \mathcal{P}(\pi^{-1}(p))$.

Definition 1.5. *Given a measure $\nu \in \mathcal{P}(X)$ we can disintegrate ν with respect to the map π. The result is a collection of measures $\{\nu_p\}_{p \in \mathrm{Gr}_2(\mathbb{R}^3)} \subset \mathcal{P}(\pi^{-1}(p))$ and a measure $\bar{\nu} := \pi_* \nu \in \mathcal{P}(X)$ such that*

$$\nu = \int_{\mathrm{Gr}_2(\mathbb{R}^3)} \nu_p \mathrm{d}\bar{\nu}\,(p) \in \mathcal{P}(X).$$

- *When $\nu_p = m_p$ for $\bar{\nu}$ almost any $p \in \mathrm{Gr}_2(\mathbb{R}^3)$ we say that ν is the natural lift of $\bar{\nu}$.*
- *In contrast, if there exists $k \in \mathbb{N}$ such that ν_p is a uniform measure supported on a set of size k for all $p \in \mathrm{Gr}_2(\mathbb{R}^3)$, then we say that ν is a k-extension of $\bar{\nu}$.*
- *We will also say that ν is a finite extension of $\bar{\nu}$ if it is a k-extension of $\bar{\nu}$ for some $k \in \mathbb{N}$ which we do not specify.*
- *We also recall that given $\mu \in \mathcal{P}(G)$, ν is said to be a measure preserving extension of $\bar{\nu}$ if $g\nu_p = \nu_{gp}$ for μ-almost every $g \in G$ and $\bar{\nu}$-almost every $p \in \mathrm{Gr}_2(\mathbb{R}^3)$.*

Since π is G-equivariant, given $\mu \in \mathcal{P}(G)$ and $\nu \in \mathcal{P}_\mu(X)$, the push-forward $\pi_*\nu$ belongs to $\mathcal{P}_\mu(\mathrm{Gr}_2(\mathbb{R}^3))$. As noted earlier, we will only consider cases when Γ_μ acts strongly irreducibly and proximally on $\mathbb{R}^3$ so we can conclude that $\pi_*\nu = \bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ is the Furstenberg measure. Our main result regarding Case I is the following.

Theorem 1.6. *Let μ be a compactly supported measure whose support generates a Zariski dense subgroup of G. Then the natural lift of the Furstenberg measure on $\mathrm{Gr}_2(\mathbb{R}^3)$ is the unique μ-stationary measure on X. Furthermore, for any $x \in X$ we have that:*

(1) The sequence $\frac{1}{n}\sum_{k=1}^{n} \mu^{*k} * \delta_x$ converges to the natural lift.
(2) For $\mu^{\otimes \mathbb{N}}$-almost every $(g_1, g_2, \dots) \in G^{\mathbb{N}}$ the sequence $\frac{1}{n}\sum_{k=1}^{n} \delta_{g_k \cdots g_1 x}$ converges to the natural lift.

The second part of Theorem 1.6 has the following immediate corollary.

Corollary 1.7. *Let Γ be a finitely generated discrete Zariski dense subgroup of G. Then the pre-image $\pi^{-1}(\operatorname{supp} \bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)})$ is the unique Γ-minimal subset in X.*

Note that a non-discrete Zariski dense subgroup of G is automatically dense in G and thus the corollary is trivial for such groups because G acts transitively on X.

Theorem 1.6 should be compared with the main result of [BQ11] which is an analogous statement. The reason that the results of Benoist and Quint fall short of being applicable to our discussion is that X is not obtained as a quotient of a Lie group by a lattice but rather by a closed group with non-trivial connected component.

In Case II we also have the following result.

Theorem 1.8. *Let μ be a compactly supported probability measure on $\mathrm{SO}(Q)(\mathbb{R})$ satisfying either one of the following:*

(a) The group generated by the support of μ is discrete and Zariski dense in $\mathrm{SO}(Q)(\mathbb{R})$.
(b) The measure μ is absolutely continuous with respect to the Haar measure on $\mathrm{SO}(Q)(\mathbb{R})$ and contains the identity in the interior of its support.

Then if ν is a μ-ergodic μ-stationary measure on X then either it is the natural lift or it is a measure preserving finite extension of the Furstenberg measure on $\mathrm{Gr}_2(\mathbb{R}^3)$.
Furthermore, for any $x \in X$ we have that:

(1) Any weak- accumulation point of the sequence $\frac{1}{n}\sum_{k=1}^{n} \mu^{*k} * \delta_x$ is a μ-stationary probability measure on X.*
(2) For $\mu^{\otimes \mathbb{N}}$-almost every $(g_1, g_2, \dots) \in G^{\mathbb{N}}$ any weak- accumulation point of the sequence $\frac{1}{n}\sum_{k=1}^{n} \delta_{g_k \cdots g_1 x}$ is a μ-stationary probability measure.*

Remark 1.9. *In fact, in the proof of Theorem 1.8 we will see that in the case μ satisfies assumption (b) the existence of a finite extension is excluded and the natural lift is the unique μ-stationary measure. See the last paragraph of §4. This implies that the second part of Theorem 1.8 yields a statement similar to Theorem 1.6.*

In Theorem 1.8 the assumptions about the measure (a) and (b) are there to ensure that $(\mathrm{Gr}_2(\mathbb{R}^3), \bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)})$ is the Poisson boundary of (Γ_μ, μ) which is the actual assumption needed for the part of the proof appearing in §4. See [Fur02, Theorem 2.17, Theorem 2.21] and also [Fur63b, Theorem 5.3].

The existence of finite extensions is analogous to the existence of atomic stationary measures in the work of Benoist and Quint. It seems to us that in many cases the existence of finite extensions can be excluded due to algebraic reasons.

The lack of uniqueness in the classification part of Theorem 1.8 is what makes the conclusion regarding distributional properties of individual orbits weaker than that in Theorem 1.6. It is not clear to us if one should expect individual orbits to equidistribute with respect to a single ergodic stationary measure or not (see Problem 1.13).

1.3. **Embedding of the Poisson boundary in X.** In this subsection we work under the assumption that we are in Case II the quadratic form Q is as in equation (1.1) and we set $H = \mathrm{SO}(Q)(\mathbb{R})$. For a long time we thought we could prove that the natural lift of the Furstenberg measure is the unique μ-stationary measure in the setting of Theorem 1.8. As Conjecture 1.1 follows from such a statement, we announced Conjecture 1.1 as a theorem in several talks and research proposals. A gap in the proof was pointed out to us by Lindenstrauss and after several failed attempts to close it we ran a computer experiment and immediately found an example of a 1-extension (see Theorem 1.10). All the other examples that we can find are obtained from this example by means of finite index and we do not understand to what extent these objects are rare and what kind of structure they possess. See Problem 1.13 and Remark 1.12. We now describe this simple example and urge the reader to ponder it as we find it mind boggling.

In the following discussion and in Figure 3 we will use the notation:

$$u^+(t) := \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad u^-(t) := \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{pmatrix} \text{ and } k := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Also, let $u_\pm := u^\pm(2)$ and note that $\Gamma_0 := \langle u_+, u_- \rangle$ is a finite index subgroup in the arithmetic group $\mathrm{SO}(Q)(\mathbb{Z})$. Hence it follows from the Borel Harish-Chandra Theorem [BHC62] and the fact that Q is defined over $\mathbb{Q}$ that Γ_0 is a lattice in H. Let us denote also by $\mathcal{C} \subset \mathrm{Gr}_2(\mathbb{R}^3)$ the *circle of isotropic planes*; that is, the set of planes $p \in \mathrm{Gr}_2(\mathbb{R}^3)$ such that there exists $v \in p \setminus \{0\}$ such that $Q(v) = 0$. Note that $\mathcal{C}$ is the unique H-invariant closed minimal subset of $\mathrm{Gr}_2(\mathbb{R}^3)$ and $\mathcal{C} = Hp_0$ where $p_0 := \mathrm{span}_\mathbb{R}(\{e_1, e_2\})$. Since $\mathrm{Stab}_H(p_0) := P$ is a minimal parabolic subgroup of H, one can also think of $\mathcal{C}$ as the full flag variety of H. If $\mu \in \mathcal{P}(H)$ is such that Γ_μ is Zariski dense in H then its Furstenberg measure is supported on the circle of isotropic planes [BQ16, §4].

Theorem 1.10. *There exists a continuous Γ_0-equivariant section $\zeta : \mathcal{C} \to X$ (i.e. $\pi(\zeta(p)) = p$ for all $p \in \mathcal{C}$). In particular, if $\mu \in \mathcal{P}(G)$ satisfies $\Gamma_\mu = \Gamma_0$, then $\zeta_* \bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ is a μ-stationary 1-extension of the Furstenberg measure of μ on $\mathrm{Gr}_2(\mathbb{R}^3)$.*

Proof. For $t \in \mathbb{R}$ we define

$$\Lambda_t := \mathrm{span}_\mathbb{Z}(\{e_1 + te_2, e_2 + 2te_3\}) \quad \text{and} \quad \Lambda_\infty := \mathrm{span}_\mathbb{Z}(\{e_2, 2e_3\}).$$

Note that $\lim_{t\to\pm\infty}[\Lambda_t] = [\Lambda_\infty] \in X$. Consider the map $\psi : \mathbb{R} \cup \{\infty\} \to X$ given by $\psi(t) = [\Lambda_t]$. Let

$$g_1 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad g_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

There is an action $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathbb{R} \cup \{\infty\}$ by fractional linear transformations, obtained by identifying $\mathbb{P}\mathbb{R}^2$ with $\mathbb{R} \cup \{\infty\}$. We claim that

$$\psi \circ g_1 = u_+ \circ \psi \quad \text{and} \quad \psi \circ g_2 = u_- \circ \psi. \tag{1.2}$$

To show this we compute

$$\begin{aligned} u_+[\Lambda_t] &= [\mathrm{span}_\mathbb{Z}(\{(1+2t)e_1 + te_2, (2+4t)e_1 + (1+4t)e_2 + 2te_3\})] \\ &= [\mathrm{span}_\mathbb{Z}(\{(1+2t)e_1 + te_2, (1+2t)e_2 + 2te_3\})] \\ &= [\Lambda_{g_1 t}] \end{aligned}$$

and

$$\begin{aligned} u_-[\Lambda_t] &= [\mathrm{span}_{\mathbb{Z}}(\{e_1 + (2+t)e_2 + (2+2t)e_3, e_2 + (2+2t)e_3\})] \\ &= [\mathrm{span}_{\mathbb{Z}}(\{e_1 + (1+t)e_2, e_2 + (2+2t)e_3\})] \\ &= [\Lambda_{g_2 t}] \end{aligned}$$

as required. Similar calculations also show that the above equalities hold true when $t = \infty$ and so (1.2) is verified.

Since $\langle g_1, g_2\rangle$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ its action on $\mathbb{PR}^2$ is minimal[1] It follows from the equivariance of π and (1.2) that $\pi\circ\psi(\mathbb{PR}^2)$ is a closed minimal Γ_0-invariant set in $\mathrm{Gr}_2(\mathbb{R}^3)$. It is thus equal to $\mathcal{C}$ since the latter is the unique such set. Moreover, it is straightforward to check that $\pi\circ\psi$ is 1-1 which shows that there exists a continuous inverse $(\pi\circ\psi)^{-1} : \mathcal{C} \to \mathbb{PR}^2$. We then define $\zeta := \psi\circ(\pi\circ\psi)^{-1} : \mathcal{C} \to X$ and note that from what we established so far it is clear that ζ is a Γ_0-equivariant. □

Remark 1.11. *The remarkable feature of the section ζ from Theorem 1.10 is that it is Γ_0-equivariant and not H-equivariant. Its image $\widetilde{\mathcal{C}} := \zeta(\mathcal{C})$ is a Γ_0-invariant circle which intersects each fibre above the circle of isotropic planes in a single 2-lattice. See Figure 2 for an illustration of the (lift of the) projection of $\widetilde{\mathcal{C}}$ to $\mathrm{PSO}_2(\mathbb{R})\backslash X_2$. Since H acts minimally on $\pi^{-1}(\mathcal{C})$, $\widetilde{\mathcal{C}}$ is not H-invariant. This minimality is one of the reasons we did not expect the existence of the section ζ.*

Remark 1.12. *After presenting the above example to Uri Bader, he managed to explain it in a conceptual manner. It seems likely that his insights could be used to resolve some of the problems presented in this paper. We expect this to be the subject of future work.*

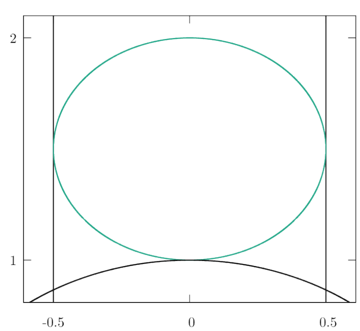

FIGURE 2. Plot of the (lift of the) projection of the Γ_0-invariant set $\widetilde{\mathcal{C}} = \zeta(\mathcal{C}) \subset X$ in $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H} \simeq \mathrm{PSO}_2(\mathbb{R})\backslash X_2$.

As this introduction is quite long, we do not dwell on the comparison between the results here and similar classification results of stationary measures on homogeneous spaces. Nevertheless, this comparison is essential if one wishes to shape a reasonable set of expectations regarding stationary measures and closed invariant sets of semisimple groups acting on spaces such as X. In particular, in the case where the acting measure generates a Zariski dense subgroup in a semisimple group, one should compare our results with

[1]This follows (for instance) from that fact that any parabolic subgroup of $\mathrm{SL}_2(\mathbb{R})$ acts minimally on $\mathrm{SL}_2(\mathbb{R})/\Gamma$, for every lattice Γ in $\mathrm{SL}_2(\mathbb{R})$. See for example [DR80, Proposition 1.5].

the seminal works of Benoist and Quint [BQ11, BQ13b, BQ12, BQ13a], Eskin and Margulis [EM04], Bourgain-Furman-Lindenstrauss-Mozes [BFLM11] and [EM13]. See also Simmons and Weiss [SW16] for results pertaining to non-semisimple Zariski closures. Compare also, the more classical results regarding measures on projective spaces originating from the seminal work of Furstenberg [Fur63a, Fur71], Furstenberg-Kesten [FK60] and Furstenberg-Kifer [FK83]. For potential applications of such classification results see [SW16]. In the opposite case when the acting measure has certain smoothness properties one can juxtapose our results with those of Nevo and Zimmer [NZ02b, NZ02a].

We wish to stress, as this cannot be stressed enough, that we follow closely the exposition and methods developed in [BQ13b]. Our main work was to overcome technical difficulties arising from the fact that X is obtained as a quotient by a group with a non-trivial connected component. Other than that we mainly needed to downgrade the generality of their discussion and hopefully maintain the quality of presentation.

In future work we plan to generalise the results of this paper and analyse actions of discrete groups on spaces with features similar to X. These include the space of homothety classes of lattices in k-planes in $\mathbb{R}^n$ but more generally bundles over projective spaces with fibres obtained as quotients of a Lie group by a lattice.

We conclude this introduction by stating some natural open problems and presenting figures pertaining to Case II.

Problem 1.13. *Let $\mu \in \mathcal{P}(H)$ be a finitely supported measure such that Γ_μ is Zariski dense in $H = \mathrm{SO}(Q)(\mathbb{R})$.*

(1) Is it true that if Γ_μ is dense in H, or if Γ_μ is cocompact in H, then the natural lift of the Furstenberg measure is the unique μ-stationary measure on X?
(2) If k_i is a sequence of natural numbers such that $k_i \to \infty$ and $\nu_i \in \mathcal{P}(X)$ is a μ-stationary k_i-extension of the Furstenberg measure on $\mathrm{Gr}_2(\mathbb{R}^3)$ is it true that ν_i converges to the natural lift of the Furstenberg measure?
(3) For $x \in X$, does the set of accumulation points $\overline{\Gamma_\mu x} \smallsetminus \Gamma_\mu x$ of the orbit $\Gamma_\mu x$ support a μ-ergodic μ-stationary probability measure?
(4) Is it true that for any $x \in X$ the sequence $\frac{1}{n}\sum_{k=1}^{n} \mu^{*k} * \delta_x$ converges to a μ-ergodic μ-stationary probability measure?
(5) Is it true that for any $x \in X$ and $\mu^{\otimes\mathbb{N}}$-almost every $(g_1, g_2, \dots) \in G^{\mathbb{N}}$ that the sequence $\frac{1}{n}\sum_{k=1}^{n} \delta_{g_k \cdots g_1 x}$ converges to a μ-ergodic μ-stationary probability measure?
(6) Is it true that if ν is a μ-ergodic μ-stationary probability measure on X which is a k-extension of $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$, then there exists a copy of the circle $\widetilde{\mathcal{C}} \subset X$ such that $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ is a covering map of degree k and $\nu(\widetilde{\mathcal{C}}) = 1$?

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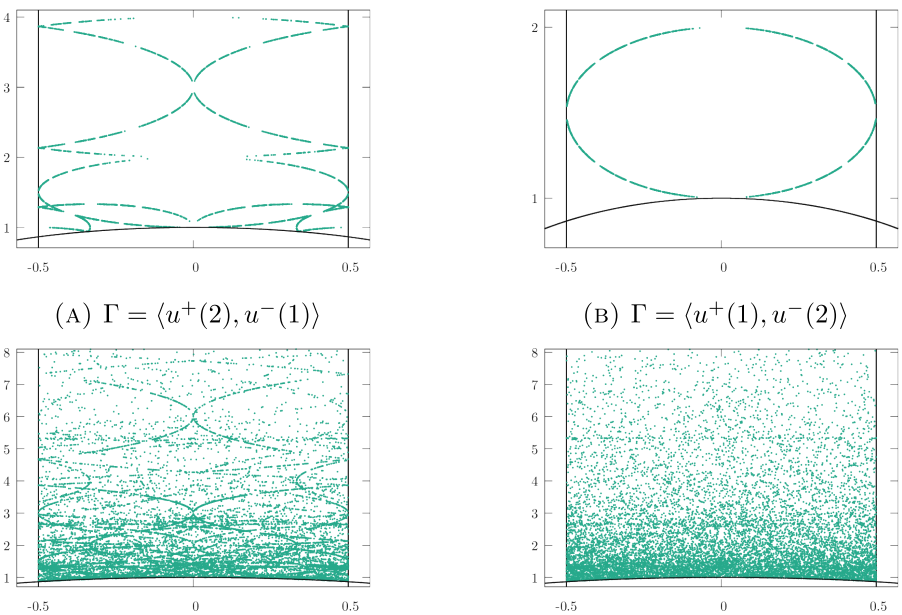
2. Generalities

Throughout the paper $\mu \in \mathcal{P}(G)$ is compactly generated, $\Gamma := \langle \operatorname{supp} \mu \rangle$ is the group generated by its support and H is the Zariski closure of Γ. Furthermore, we assume we are



(A) $\Gamma = \langle u^{+}(2), u^{-}(1)\rangle$ (B) $\Gamma = \langle u^{+}(1), u^{-}(2)\rangle$

(C) $\Gamma = \langle u^{+}(2), u^{-}(1), k\rangle$ (D) $\Gamma = \langle u^{+}(1), u^{-}(1)\rangle$

FIGURE 3. Plots in $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ of the projections of random points in the Γ-orbit of $[\Lambda_0]$ for various choices of Γ.

either in Case I or Case II. Given $k, l \in \mathbb{N}$ with $k < l$ and elements $b_k, \dots, b_l \in G$ we use the following notation to denote products

$$b_k^l := b_k \cdots b_l \quad \text{and} \quad b_l^k := b_l \cdots b_k.$$

2.1. **A restatement and the structure of the paper.** For convenience of reference we aim to state a unified theorem whose statement captures both Theorem 1.6 and Theorem 1.8. In order to do so we need to define some more objects. For details regarding the following facts we refer the reader to [BQ16, §2.5]. Let $A := \operatorname{supp}\mu$ and $B := A^{\mathbb{N}}$ be the space of infinite sequences indexed by the positive integers. Let $\beta := \mu^{\otimes\mathbb{N}}$ be the Bernoulli measure and $S : B \to B$ be the shift map $Sb = (b_2, b_3, \dots)$, where $b = (b_1, b_2, \dots)$. Given $\nu \in \mathcal{P}_\mu(X)$ it is well known that for β-almost every $b \in B$ the sequence $b_1^n\nu$ converges to a probability measure denoted ν_b known as the *limit measure* of ν with respect to b. Hence, the map $b \mapsto \nu_b$ is almost surely well defined and equivariant in the sense that $\nu_b = b_1\nu_{Sb}$ for β-almost every $b \in B$. Moreover, one can recover the measure ν by integrating

$$\nu = \int_B \nu_b \mathrm{d}\beta\,(b)\,.$$

The following theorem is our unified statement and the reader can readily check that Theorems 1.6 and 1.8 follow from it.

Theorem 2.1. *Let $\mu \in \mathcal{P}(G)$ be a compactly supported measure and suppose we are in Case I or Case II. Let $\nu \in \mathcal{P}_\mu(X)$ be μ-ergodic.*

(a) If for β-almost every $b \in B$ the limit measure ν_b is non-atomic, then ν is the natural lift of the Furstenberg measure of μ on $\mathrm{Gr}_2(\mathbb{R}^3)$.
(b) In Case I it holds that for β-almost every $b \in B$ the limit measure ν_b is non-atomic.

(c) *In Case II, if it does not hold that for β-almost every $b \in B$ the measure ν_b is non-atomic and if Γ is discrete or if μ is absolutely continuous with respect to the Haar measure on H and contains the identity in the interior of its support, then ν is a measure preserving finite extension of the Furstenberg measure of μ on $\mathrm{Gr}_2(\mathbb{R}^3)$.*

(d) *In both Case I and Case II for any $x \in X$, any weak-* limit point of the sequence $\frac{1}{n}\sum_{k=1}^{n} \mu^{*k} * \delta_x$ is an element of $\mathcal{P}_\mu(X)$. Moreover, for β-almost every $b \in B$ any weak-* accumulation point of the sequence $\frac{1}{n}\sum_{k=1}^{n} \delta_{b_k^1 x}$ is an element of $\mathcal{P}_\mu(X)$.*

The rest of the paper is devoted to the proof of Theorem 2.1. In the rest of §2 we collect notation and results needed for the rest of the paper.

We establish part (a) of Theorem 2.1 in §3 by means of the so called *exponential drift* argument of Benoist and Quint.

We establish part (c) of Theorem 2.1 in §4. To do this we will use a result of Ledrappier [Led85] that in this case the measure space $(\mathrm{Gr}_2(\mathbb{R}^3), \bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)})$ is the Poisson boundary of (Γ, μ). We note that part (c) of the theorem must be taken into account in conjunction with Theorem 1.10 which says that this possibility is not vacuous.

We establish part (d) of Theorem 2.1 at the end of §5. Given the analysis of Benoist and Quint [BQ13a] the proof boils down to a non-escape of mass result which is proved in §5. The aim is to show that a certain function on X, which can be thought of as a height function, tends to be contracted by the random walk. This will enable us to prove that the 'cusp' in X is 'unstable' with respect to the action induced by μ.

Finally we establish part (b) of Theorem 2.1 in §6 using an argument which was developed by Benoist and Quint in [BQ13b]. The main point is to show that the diagonal in $X \times X$ is 'unstable'.

2.2. **The boundary map and other equivariant maps.** When studying μ-stationary probability measures one is naturally led to consider equivariant maps $\zeta : B \to Y$ for various spaces Y on which Γ acts. Here equivariant means that for β-almost every $b \in B$ one has $\zeta(b) = b_1\zeta(Sb)$. The reason for this is that given such an equivariant map, the measure $\nu = \zeta_*\beta$ belongs to $\mathcal{P}_\mu(Y)$ and the limit measures ν_b are equal to $\delta_{\zeta(b)}$ for β-almost every $b \in B$.

In order to proceed we must choose a minimal parabolic subgroup of H. In both Case I and Case II the subgroup of H consisting of upper triangular elements is a minimal parabolic subgroup of H. We will denote this subgroup by P. By [BQ16, Proposition 10.1] the set $\mathcal{P}_\mu(H/P)$ consists of a single measure and it is μ-proximal. This implies that there is a unique measurable equivariant map

$$\xi : B \to H/P$$

which is referred to as the *boundary map*.

The mechanism giving rise to the equivariant maps we will consider is as follows: Let V be a representation of H. If $W_0 \subseteq V$ is a subspace of dimension d which is P-invariant then there is a well defined H-equivariant map $H/P \to \mathrm{Gr}_d(V)$ defined by

$$hP \mapsto hW_0.$$

For $\eta \in H/P$ we then denote the image of η by W_η. In turn, the composition of this map with ξ gives rise to an equivariant map $B \to \mathrm{Gr}_d(V)$ given by

$$b \mapsto W_{\xi(b)}$$

for β-almost every $b \in B$. Hence, for β-almost every $b \in B$ we define $W_b := W_{\xi(b)}$.

For example, consider the representation of H on $\mathbb{R}^3$. As in §1.3, let $p_0 := \operatorname{span}_{\mathbb{R}}(\{e_1, e_2\}) \in \operatorname{Gr}_2(\mathbb{R}^3)$. In both Case I and Case II p_0 is P-invariant and therefore one obtains an equivariant map $H/P \to \operatorname{Gr}_2(\mathbb{R}^3)$ given by

$$\eta \mapsto \eta p_0 =: p_\eta.$$

Using this map in conjunction with ξ as described above gives rise to the equivariant map $B \to \operatorname{Gr}_2(\mathbb{R}^3)$ given by

$$b \mapsto p_{\xi(b)} =: p_b.$$

Hence, the push-forward of β under $b \mapsto p_b$ is a μ-stationary probability measure on $\operatorname{Gr}_2(\mathbb{R}^3)$. Since the Furstenberg measure $\bar{\nu}_{\operatorname{Gr}_2(\mathbb{R}^3)}$ is the unique such measure, we deduce that $(\bar{\nu}_{\operatorname{Gr}_2(\mathbb{R}^3)})_b = \delta_{p_b}$ for all β-almost every $b \in B$. This implies the following proposition which constitutes the first step towards classifying the μ-stationary measures on X.

Proposition 2.2. *Let $\nu \in \mathcal{P}_\mu(X)$. Then, for β-almost every $b \in B$ we have $\nu_b(\pi^{-1}(p_b)) = 1$.*

Proof. Since π is H-equivariant we have that $\pi_*\nu \in \mathcal{P}_\mu(\operatorname{Gr}_2(\mathbb{R}^3))$. Since the Furstenberg measure $\bar{\nu}_{\operatorname{Gr}_2(\mathbb{R}^3)}$ is the unique measure in $\mathcal{P}_\mu(\operatorname{Gr}_2(\mathbb{R}^3))$ we deduce that $\pi_*\nu = \bar{\nu}_{\operatorname{Gr}_2(\mathbb{R}^3)}$. Furthermore, it follows that $\pi_*\nu_b = (\bar{\nu}_{\operatorname{Gr}_2(\mathbb{R}^3)})_b$ for β-almost every $b \in B$. As we observed above, $(\bar{\nu}_{\operatorname{Gr}_2(\mathbb{R}^3)})_b = \delta_{p_b}$ for β-almost every $b \in B$ and the statement follows. □

Next we define some subgroups of G. Let $G_0 := \operatorname{Stab}_G(p_0)$ and R_0 be the solvable radical of G_0. Since $P < G_0$ and R_0 is a normal subgroup of G_0 it is clear that R_0 is invariant under conjugation by P. Moreover,

$$L_0 := \left\{ \begin{pmatrix} \alpha & * & * \\ 0 & \alpha & * \\ 0 & 0 & \alpha^{-2} \end{pmatrix} : \alpha \in \mathbb{R}^\times \right\} \supset R_0$$

is also easily seen to be invariant under conjugation by P. The P-invariance of these subgroups allows us to define equivariant maps from H/P to the set of subgroups of G. In other words, the maps

$$hP \mapsto hR_0h^{-1},\ hP \mapsto hL_0h^{-1} \text{ and } hP \mapsto hG_0h^{-1}$$

are well defined. As before, for $\eta \in H/P$ we let R_η, L_η and G_η denote the images of these maps. Combined with the map ξ, these maps allow us to define equivariant maps from B to the set of subgroups of G. These maps are given explicitly as

$$b \mapsto R_b := R_{\xi(b)},\ b \mapsto L_b := L_{\xi(b)} \text{ and } b \mapsto G_b := G_{\xi(b)}.$$

Later on we will use corresponding lower case Gothic letters to denote the Lie algebras. As R_0 is normal in L_0 we may define the 1-parameter unipotent quotient group $U_0 := L_0/R_0$. Similarly for any $\eta \in H/P$ we define the 1-parameter unipotent quotient group

$$U_\eta := L_\eta/R_\eta.$$

This assignment is clearly equivariant and again for $b \in B$ we will use the notation

$$U_b := U_{\xi(b)}. \tag{2.1}$$

It is straightforward to check that $G_b = \operatorname{Stab}_G(p_b)$. Moreover, R_b acts trivially on p_b. It follows that the action of U_b on p_b is well defined and nontrivial for β-almost every $b \in B$. The crucial point for us is that this action descends to a nontrivial action of U_b on $\pi^{-1}(p_b)$. The main step in the proof of Theorem 2.1(a) is to show that if $\nu \in \mathcal{P}_\mu(X)$ is μ-ergodic and the limit measures ν_b are non-atomic almost surely, then for β-almost every b, ν_b is U_b-invariant. In the following subsection we show that this unipotent invariance implies that ν is the natural lift of $\bar{\nu}_{\operatorname{Gr}_2(\mathbb{R}^3)}$.

2.3. **Reduction of the proof of Theorem 2.1(a).** The core of the proof of Theorem 2.1(a) is an application of the exponential drift argument of Benoist and Quint. This is an elaborate argument which takes quite a lot of apparatus. In this section we isolate the following lemma whose statement does not require any preparation and relying on this lemma we prove a proposition which reduces the proof of Theorem 2.1(a) to establishing the β-almost sure U_b-invariance of the limit measures ν_b.

Lemma 2.3. *Let $\mu \in \mathcal{P}(G)$ be compactly supported and suppose Case I or Case II holds. Then for any $\delta > 0$ and $R > 0$ there exists $n_0 > 0$ such that for any $v \in \mathbb{R}^3 \smallsetminus \{0\}$, $w \in \wedge^2\mathbb{R}^3 \smallsetminus \{0\}$ and $n > n_0$ one has*

$$\beta\left(\left\{b \in B : \frac{\|b_1^n v\|}{\|b_1^n w\|^{1/2}} < R\right\}\right) < \delta.$$

The statement of Lemma 2.3 and its use in the proof of Proposition 2.4 illustrates in a simple fashion the role played by comparison of growth rates of vectors under random products in various representations, which is a recurring theme in the paper. We note that the proof of Lemma 2.3 will only be given in §2.4 after the necessary notation and tools regarding Lyapunov exponents will be presented. During the proof of Proposition 2.4 we will need to use the following construction. Let

$$B^X := B \times X,\ \beta^X := \int_B \delta_b \otimes \nu_b \mathrm{d}\beta\,(b) \tag{2.2}$$

and $T : B^X \to B^X$ be defined by $T(b,x) := (Sb, b_1^{-1}x)$. If $\nu \in \mathcal{P}_\mu(X)$ then T preserves β^X and if ν is assumed to be μ-ergodic then T is ergodic. Following Benoist and Quint, we will call the system (B^X, T, β^X) the *backwards dynamical system*, see [BQ16, §2.5].

Proposition 2.4. *Let $\nu \in \mathcal{P}_\mu(X)$ be μ-ergodic. Suppose that for β-almost every $b \in B$ the limit measure ν_b is U_b-invariant, where U_b is as in (2.1). Then ν is the natural lift of $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$.*

Proof. Below we will show that the almost sure U_b-invariance of the ν_b's together with Lemma 2.3 imply that $\nu_b = m_{p_b}$ for β-almost every $b \in B$. This will finish the proof because

$$\nu = \int_B \nu_b \mathrm{d}\beta\,(b) = \int_B m_{p_b} \mathrm{d}\beta\,(b) = \int_{\mathrm{Gr}_2(\mathbb{R}^3)} m_p \mathrm{d}\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}\,(p)\,,$$

where the last equality follows because the Furstenberg measure $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ is the pushforward of β under $b \mapsto p_b$.

Assume that ν_b is U_b-invariant β-almost surely. By Proposition 2.2, ν_b is supported on $\pi^{-1}(p_b)$ for β-almost every $b \in B$. The classification of U_b-invariant measures on $\pi^{-1}(p_b)$ due to Dani [Dan78, Theorem A] (see also [Rat91]) implies that $\nu_b = t_b m_b + (1-t_b)\eta_b$ for some $0 \le t_b \le 1$, where η_b is a U_b-invariant measure supported on the collection of periodic U_b-orbits in $\pi^{-1}(p_b)$. The equivariance of the ν_b's and the m_b's imply the equivariance of the η_b's which in turn implies that $t_b = t_{Sb}$ for β-almost every $b \in B$. The ergodicity of the shift map implies that $t_b = t$ is β-almost surely constant and then the ergodicity of ν implies that either $t = 0$ or $t = 1$. We assume that $t = 0$; that is that β-almost surely ν_b is supported purely on periodic U_b-orbits and derive a contradiction. This assumption may be restated in the backwards dynamical system as follows: Let

$$\Sigma := \{(b, [\Lambda]) \in B^X : [\Lambda] \in \pi^{-1}(p_b),\ \text{and } U_b[\Lambda] \text{ is periodic}\}.$$

Our assumption that $t = 0$ implies that $\beta^X(\Sigma) = 1$.

For a 2-lattice Λ we let $|\Lambda|$ denote the covolume of Λ. Recall that a 2-lattice $[\Lambda] \in \pi^{-1}(p_b)$ has a periodic U_b-orbit if and only if Λ intersects the eigenline ℓ_b of U_b in the plane p_b non-trivially. Therefore, the function

$$\rho : \Sigma \to (0, \infty), \text{ given by } \rho(b, [\Lambda]) := \frac{|\Lambda \cap \ell_p|}{|\Lambda|^{1/2}}$$

is well defined β^X-almost surely. Choose $R > 0$ so that the pre-image $\Sigma_R := \rho^{-1}((0, R))$ satisfies $\beta^X(\Sigma_R) > 1/2$.

Note that, if $(b, [\Lambda]) \in \Sigma$, then choosing a primitive vector $v \in \Lambda \cap \ell_b$ and a basis u_1, u_2 of Λ, if we set $w = u_1 \wedge u_2 \in \wedge^2\mathbb{R}^3$ then we have that

$$\rho(b, [\Lambda]) = \frac{\|v\|}{\|w\|^{1/2}}.$$

It follows from the equivariance that for $\mu^{\otimes n}$-almost every $a \in G^n$ and β^X-almost every $(b, [\Lambda]) \in \Sigma$ we have $\ell_{ab} = a_1^n \ell_b$ and so $a_1^n v$ is a primitive vector in $\ell_{ab} \cap a_1^n \Lambda$ and $a_1^n u_i, i = 1, 2$ is a basis for $a_1^n \Lambda$. We conclude that

$$\rho(ab, a_1^n[\Lambda]) = \frac{\|a_1^n v\|}{\|a_1^n w\|^{1/2}}. \tag{2.3}$$

Consider the operator A defined by

$$\mathrm{A}\, f(b, [\Lambda]) := \int_G f(gb, g[\Lambda]) \mathrm{d}\mu\,(g) \quad \text{for} \quad f : B^X \to [0, \infty) \text{ measurable.}$$

We take $f := \mathbf{1}_{\Sigma_R}$. Lemma 2.3 and (2.3) imply that for β^X-almost every $(b, [\Lambda]) \in \Sigma$ we have

$$\lim_{n\to\infty} \mathrm{A}^n\, f(b, [\Lambda]) = \lim_{n\to\infty} \int_{G^n} \mathbf{1}_{\Sigma_R}(ab, a_1^n[\Lambda]) \mathrm{d}\mu^{\otimes n}\,(a) = 0. \tag{2.4}$$

It is easy to check that the operator A preserves the measure β^X. Hence for any $n \in \mathbb{N}$ we have

$$1/2 \le \int_{B\times X} \mathbf{1}_{\Sigma_R} \mathrm{d}\beta^X = \int_{B\times X} \mathrm{A}^n\, \mathbf{1}_{\Sigma_R} \mathrm{d}\beta^X.$$

But on the other hand using (2.4) and Lebesgue's dominated convergence theorem we see that the right hand side of the above equation tends to 0, which is a contradiction as required. □

2.4. **The Iwasawa cocycle.** Let H be as in Case I or Case II and let $H = KP$ be an Iwasawa decomposition of H where P is as in §2.2 and K is the maximal compact subgroup of H corresponding to the inner product coming from the standard basis $\{e_1, e_2, e_3\}$. Let $\mathfrak{z}$ be the maximal abelian subspace of the Lie algebra of P and define $Z = \{\exp(z) : z \in \mathfrak{z}\}$ to be the corresponding Cartan subgroup of H. We denote by $\log : Z \to \mathfrak{z}$ the inverse of exp. Moreover, we set N to be the unipotent radical of P so that $P = ZN$. See [Kna02] for details.

Let $s : H/P \to H/N$ be a measurable section with image in KN. For $h \in H$ and $\eta \in H/P$ let $\alpha : H \times H/P \to Z$ be defined so that $\alpha(h, \eta)$ is the unique element of Z such that

$$hs(\eta) = s(h\eta)\, \alpha(h, \eta). \tag{2.5}$$

Note that since Z normalises N it acts on H/N from the right and moreover this action is transitive with trivial stabilisers, so equation (2.5) makes sense and defines $\alpha(h,\eta)$ uniquely. The *Iwasawa cocycle* is the map

$$\sigma(h,\eta) := \log \alpha(h,\eta).$$

Indeed, it is not hard to see from (2.5) that the cocycle relations $\alpha(gh,\eta) = \alpha(g,h\eta)\alpha(h,\eta)$, $\sigma(gh,\eta) = \sigma(g,h\eta) + \sigma(h,\eta)$ hold. See [BQ16, §8.2] for details.

Let $\mathrm{E} : B \to Z$ be given by

$$\mathrm{E}(b) := \alpha(b_1, \xi(Sb))$$

and for $n \in \mathbb{N}$ let $\mathrm{E}_n(b) : B \to Z$ be

$$\mathrm{E}_n(b) := \prod_{i=1}^{n} \mathrm{E}(S^{i-1}b) = \alpha(b_1^n, \xi(S^n b)).$$

Additionally, we define the corresponding logarithmic versions $\mathrm{L} : B \to \mathfrak{z}$ and $\mathrm{L}_n : B \to \mathfrak{z}$ as

$$\mathrm{L}(b) := \log \mathrm{E}(b) \quad \text{and} \quad \mathrm{L}_n(b) := \log \mathrm{E}_n(b).$$

Let V be a finite dimensional representation of H and let $\mathcal{W}_{\mathfrak{z}}(V)$ be the set of weights of V relative to $\mathfrak{z}$. For $\omega \in \mathcal{W}_{\mathfrak{z}}(V)$ we will use V^{ω} to denote the corresponding weight space. Let $\mathcal{H}_{\mathfrak{z}}(V)$ be the set of highest weights of the representation V. For $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ we write $V[\omega]$ for the corresponding isotypic component. Note that $(V[\omega])^{\omega}$ is P-invariant and pointwise fixed by N.

By the discussion in §2.2, the P-invariant subspace $(V[\omega])^{\omega}$ gives rise to an equivariant map from H/P to the set of subspaces of V given by

$$gP \mapsto g(V[\omega])^{\omega}.$$

To reduce the notational clutter, for $\eta = gP \in H/P$ we simplify the notation to

$$V_{\eta}[\omega] := g\left(V[\omega]\right)^{\omega}.$$

For $b \in B$ we will also use the notation

$$V_b[\omega] := V_{\xi(b)}[\omega].$$

For $\omega \in \mathcal{W}_{\mathfrak{z}}(V)$, let $\chi^{\omega} : Z \to \mathbb{R}^{\times}$ be the character corresponding to the weight ω, that is

$$\chi^{\omega} := \exp \circ \omega \circ \log.$$

We will use the following lemma on multiple occasions. It is the same as [BQ11, Lemma 5.4] except that we replaced "irreducible representation" by "isotypic component".

Lemma 2.5. *Suppose V is a representation of G. Then, for β-almost every $b \in B$ and for all $n \in \mathbb{N}$, $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ and $v \in V_{S^n b}[\omega]$ we have*

$$\chi^{\omega}(\mathrm{E}_n(b)) = \|b_1^n v\| / \|v\|.$$

2.5. **The Lyapunov vector and pairs of highest weights.** Our choice of P in §2.2 implies that $\mathfrak{z}$ consists of diagonal traceless matrices. Let $\mathfrak{z}^+$ be the Weyl chamber associated with P and $\mathfrak{z}^{++}$ denote its interior.

- In Case I we have $\mathfrak{z}^{++} = \{\mathrm{diag}(t_1, t_2, t_3) \in \mathfrak{z} : t_1 > t_2 > t_3\}$.
- In Case II we have $\mathfrak{z}^{++} = \{\mathrm{diag}(t, 0, -t) \in \mathfrak{z} : t > 0\}$.

To keep a unified treatment we will denote elements in $\mathfrak{z}^{++}$ by $\mathrm{diag}(t_1, t_2, t_3)$ and use the inequalities $t_1 > t_2 > t_3$ which are valid in both cases. The following theorem is a collection of relevant statements regarding the Lyapunov vector of μ. In [BQ16] this collection of statements is referred to as "the law of large numbers on H".

Theorem 2.6. *Let*

$$\sigma_\mu := \int_{H\times H/P} \sigma \mathrm{d}\mu \mathrm{d}\bar{\nu}_{H/P},$$

where $\bar{\nu}_{H/P} \in \mathcal{P}_\mu(H/P)$ is the unique μ-stationary probability measure on H/P. Then:

(1) [BQ16, Theorem 10.9(a)] *The Iwasawa cocycle σ is integrable and hence $\sigma_\mu \in \mathfrak{z}$ is well defined.*
(2) [BQ16, Theorem 10.9(a)] *For β-almost every $b \in B$ we have $\lim_{n\to\infty} \frac{1}{n} \mathrm{L}_n(b) = \sigma_\mu$.*
(3) [BQ16, Theorem 4.28(b) + Corollary 10.12] *If V is an irreducible representation of H with highest weight ω, then for all $v \in V \smallsetminus \{0\}$ and for β-almost every $b \in B$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|b_n^1 v\|}{\|v\|} = \omega(\sigma_\mu).$$

This sequence also converges in $L^1(B, \beta)$ uniformly over $v \in V \smallsetminus \{0\}$.
(4) [BQ16, Theorem 10.9(f)] *One has $\sigma_\mu \in \mathfrak{z}^{++}$. In particular, if ω is a positive weight, then $\omega(\sigma_\mu) > 0$.*

The vector σ_μ defined in Theorem 2.6 is called the *Lyapunov vector* of μ.

There are two pairs of highest weights which play a prominent role in our discussion. The first pair consists of the highest weights of the irreducible representations of H on $\mathbb{R}^3$ and on $\wedge^2\mathbb{R}^3$ which we denote by $\omega_{\mathbb{R}^3}$ and $\omega_{\wedge^2\mathbb{R}^3}$ respectively. The important fact regarding this pair is that for $t = \operatorname{diag}(t_1, t_2, t_3) \in \mathfrak{z}$ we have $\omega_{\mathbb{R}^3}(t) = t_1$ and $\omega_{\wedge^2\mathbb{R}^3}(t) = t_1 + t_2$ so that $(\omega_{\mathbb{R}^3} - \frac{1}{2}\omega_{\wedge^2\mathbb{R}^3})(t) = \frac{1}{2}(t_1 - t_2)$ and so $\omega_{\mathbb{R}^3} - \frac{1}{2}\omega_{\wedge^2\mathbb{R}^3}$ is positive. Thus, by part (4) of Theorem 2.6 we have the following fundamental inequality: In both of Case I and Case II one has

$$\omega_{\mathbb{R}^3}(\sigma_\mu) - \frac{1}{2}\omega_{\wedge^2\mathbb{R}^3}(\sigma_\mu) > 0. \tag{2.6}$$

Equipped with this inequality and with Theorem 2.6 we can now easily deduce Lemma 2.3 which played an important role in the proof of Proposition 2.4.

Proof of Lemma 2.3. Given $\epsilon > 0$, it follows from part (3) of Theorem 2.6 that there exists $n_0 > 0$ such that for all $v \in \mathbb{R}^3 \smallsetminus \{0\}$, $w \in \wedge^2\mathbb{R}^3 \smallsetminus \{0\}$ and $n > n_0$ one has

$$\mu^{*n}\left(\left\{ g \in G : \begin{array}{l} \|gv\| > \|v\| \exp(n(\omega_{\mathbb{R}^3}(\sigma_\mu) - \epsilon)) \\ \|gw\| < \|w\| \exp(n(\omega_{\wedge^2\mathbb{R}^3}(\sigma_\mu) + \epsilon)) \end{array} \right\}\right) > 1 - \epsilon.$$

By (2.6), on choosing ϵ so that $\omega_{\mathbb{R}^3}(\sigma_\mu) - \frac{1}{2}\omega_{\wedge^2\mathbb{R}^3}(\sigma_\mu) - \frac{3}{2}\epsilon = \epsilon' > 0$ we get that for all for all $v \in \mathbb{R}^3 \smallsetminus \{0\}$, $w \in \wedge^2\mathbb{R}^3 \smallsetminus \{0\}$ and $n > n_0$,

$$\mu^{*n}\left(\left\{ g \in G : \frac{\|gv\|}{\|gw\|^{1/2}} > \frac{\|v\|}{\|w\|^{1/2}} \exp(n\epsilon') \right\}\right) > 1 - \epsilon.$$

The statement of the lemma now readily follows. □

We now discuss the second pair of highest weights that will concern us. Let $\mathfrak{r}_0$ and $\mathfrak{l}_0$ be the Lie algebras of the Lie groups R_0 and L_0 we defined in §2.2. The Lie algebras $\mathfrak{r}_0$ and $\mathfrak{l}_0$ correspond to P-invariant lines in the representations $\wedge^3\mathfrak{g}$ and $\wedge^4\mathfrak{g}$ respectively. We denote the corresponding weights by $\omega_{\mathfrak{r}_0} \in \mathcal{H}_{\mathfrak{z}}(\wedge^3\mathfrak{g})$ and $\omega_{\mathfrak{l}_0} \in \mathcal{H}_{\mathfrak{z}}(\wedge^4\mathfrak{g})$. Given $t = \operatorname{diag}(t_1, t_2, t_3) \in \mathfrak{z}$ we have $\omega_{\mathfrak{l}_0}(t) = 2(t_1 - t_3)$ and $\omega_{\mathfrak{r}_0}(t) = t_1 + t_2 - 2t_3$ so that $(\omega_{\mathfrak{l}_0} - \omega_{\mathfrak{r}_0})(t) = t_1 - t_2$ and so $\omega_{\mathfrak{l}_0} - \omega_{\mathfrak{r}_0}$ is positive. By part (4) of Theorem 2.6 we arrive at the following fundamental inequality: In both Case I and Case II

$$\omega_{\mathfrak{l}_0}(\sigma_\mu) - \omega_{\mathfrak{r}_0}(\sigma_\mu) > 0. \tag{2.7}$$

We will often work with the difference $\omega_{\mathfrak{l}_0} - \omega_{\mathfrak{r}_0}$ and use the notation

$$\omega_{\mathfrak{l}_0/\mathfrak{r}_0} := \omega_{\mathfrak{l}_0} - \omega_{\mathfrak{r}_0} \quad \text{and} \quad \chi_{\mathfrak{l}_0/\mathfrak{r}_0} := \exp \circ \omega_{\mathfrak{l}_0/\mathfrak{r}_0} \circ \log . \tag{2.8}$$

2.6. **Two lemmas about representations.** In this subsection we collect some representation theoretic results which are specific to the representations we are interested in. Let V be a representation of H and let $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$. We denote by $\tau_\omega : V \to V[\omega]$ the natural projection and note that it is H-equivariant. We always assume that the norm we choose on a vector space V is induced by an inner product with respect to which the isotypic components are orthogonal and such that the maximal compact $K < H$ from the Iwasawa decomposition acts by isometries. For $\alpha > 0$ we denote

$$V_{\triangleleft\alpha}[\omega] := \{v \in V : \|\tau_\omega(v)\| \geq \alpha \|v\|\}. \tag{2.9}$$

This is the complement of a projective neighbourhood of $\ker \tau_\omega$.

For $p \in \mathrm{Gr}_2(\mathbb{R}^3)$ we define $G_p := \mathrm{Stab}_G(p)$ and R_p to be the radical of G_p. Since $G_{p_b} = G_b$ for β-almost every $b \in B$ these definitions are compatible with our previous definitions from §2.2. As usual, we denote the corresponding Lie algebras by lower-case Gothic letters. Moreover in Case II, also recall the notation $\mathcal{C}$ for the circle of isotropic planes in $\mathrm{Gr}_2(\mathbb{R}^3)$ that we introduced in §1.3.

Lemma 2.7. *The following hold:*

(1) The weight $\omega_{\mathfrak{l}_0}$ is a maximal weight in $\mathcal{H}_{\mathfrak{z}}(\wedge^4\mathfrak{g})$.

(2) In Case I there exists $\alpha > 0$ such that for all $p \in \mathrm{Gr}_2(\mathbb{R}^3)$, $u \in \wedge^3\mathfrak{r}_p$ and $v \in \mathfrak{g}$,

$$v \wedge u \in (\wedge^4\mathfrak{g})_{\triangleleft\alpha}[\omega_{\mathfrak{l}_0}].$$

(3) In Case II there exists $\alpha > 0$ such that for all $p \in \mathcal{C}$, $u \in \wedge^3\mathfrak{r}_p$ and $v \in \mathfrak{g}_p$,

$$v \wedge u \in (\wedge^4\mathfrak{g})_{\triangleleft\alpha}[\omega_{\mathfrak{l}_0}].$$

(4) For all $\eta \in H/P$ one has

$$\{v \wedge u : v \in \mathfrak{g}, u \in \wedge^3\mathfrak{r}_\eta\} \cap (\wedge^4\mathfrak{g})_\eta[\omega_{\mathfrak{l}_0}] = \wedge^4\mathfrak{l}_\eta. \tag{2.10}$$

Proof. First we prove (1). Given $t = \mathrm{diag}\,(t_1, t_2, t_3) \in \mathfrak{z}$ the eigenvalues of ad_t on $\mathfrak{g}$ determine its eigenvalues on $\wedge^4\mathfrak{g}$. Namely, they are all possible sums of 4 eigenvalues of ad_t on $\mathfrak{g}$ corresponding to different eigenlines. It is then clear from the ordering of the weights of the adjoint representation that the maximal weight of the fourth wedge is $\omega_{\mathfrak{l}_0}(t) = 2(t_1 - t_3)$.

Next we prove (2). In Case I we have $K \cong \mathrm{SO}_3(\mathbb{R})$ and hence it acts transitively on $\mathrm{Gr}_2(\mathbb{R}^3)$. Write $p = kp_0$ for some $k \in K$ and then $\mathfrak{r}_p = k\mathfrak{r}_0$. Since the set $(\wedge^4\mathfrak{g})_{\triangleleft\alpha}[\omega_{\mathfrak{l}_0}]$ is K-invariant, it is enough to prove (2) for $p = p_0$. Let $\{e_{ij}\}_{1\leq i,j\leq 3}$ be the basis of unit matrices in $\mathrm{Mat}_3(\mathbb{R})$ and let $d_1 := e_{11} + e_{22} - 2e_{33}$ and $d_2 := e_{11} - e_{22}$ so that $\{d_1, d_2\} \cup \{e_{ij}\}_{1\leq i,j\leq 3, i\neq j}$ forms a basis of $\mathfrak{g}$. Since $\wedge^3\mathfrak{r}_0$ is one dimensional the collection of pure wedges $\mathfrak{w}_0 := \{v \wedge u : v \in \mathfrak{g}, u \in \wedge^3\mathfrak{r}_0\}$ forms a subspace of $\wedge^4\mathfrak{g}$. It follows that $\mathfrak{w}_0 \subset (\wedge^4\mathfrak{g})_{\triangleleft\alpha}[\omega_{\mathfrak{l}_0}]$ for some positive α provided that $\mathfrak{w}_0 \cap \ker \tau_{\omega_{\mathfrak{l}_0}} = 0$. To this end, let $u_0 = d_1 \wedge e_{23} \wedge e_{13} \in \wedge^3\mathfrak{r}_0$ and assume by way of contradiction that there exists $v_0 \in \mathfrak{g} \smallsetminus \{0\}$ such that $v_0 \wedge u_0 \in \ker \tau_{\omega_{\mathfrak{l}_0}}$. We may assume without loss of generality that v_0 is orthogonal to $\mathfrak{r}_0$ or in other words

$$v_0 := v_{21}e_{21} + v_{31}e_{31} + v_{32}e_{32} + v_{22}d_2 + v_{12}e_{12}.$$

Since the condition $v_0 \wedge u_0 \in \ker \tau_{\omega_{\mathfrak{l}_0}}$ is H-invariant and u_0 is P-invariant we get that

$$pv_0 \wedge u_0 \in \ker \tau_{\omega_{\mathfrak{l}_0}} \text{ for all } p \in P. \tag{2.11}$$

For $v \in \mathfrak{g}$ it is easy to check that
$$v \wedge u_0 \in \ker \tau_{\omega_{\mathfrak{l}_0}} \text{ if and only if } \langle v, e_{12} \rangle = 0. \tag{2.12}$$
For $t \in \mathbb{R}^2$ let $p(t) \coloneqq \exp(t_1 e_{12} + t_2 e_{13}) \in P$. It is easy (but tedious) to compute
$$\langle p(t) v_0, e_{12} \rangle = v_{12} - t_1 v_{22} - t_1^2 v_{21} + t_2 v_{32} - t_1 t_2 v_{31}.$$
Combining this computation with (2.11) and (2.12) gives that $v_0 = 0$ which is a contradiction as required.

The proof of (3) is very similar to the proof of part (2). Since we only consider $p \in \mathcal{C}$ and in this case $K \cong \mathrm{SO}_2(\mathbb{R})$ acts transitively on $\mathcal{C}$, we can reduce to the case when $p = p_0$. As before this will follow provided that $\{v \wedge u : v \in \mathfrak{g}_0, u \in \wedge^3 \mathfrak{r}_0\}$ intersects $\ker \tau_{\omega_{\mathfrak{l}_0}}$ trivially. Suppose there exists $v_0' \in \mathfrak{g}_0 \smallsetminus \mathfrak{r}_0$ such that $v_0' \wedge u_0 \in \ker \tau_{\omega_{\mathfrak{l}_0}}$. Without loss of generality we may suppose that v_0' is orthogonal to $\mathfrak{r}_0$ and hence we may write
$$v_0' := v_{12} e_{12} + v_{22} d_2 + v_{21} e_{21}.$$
For $t \in \mathbb{R}$, let $p'(t) \coloneqq \exp(t(e_{12} + e_{23})) \in P$. Another tedious computation reveals that
$$\langle p'(t) v_0', e_{12} \rangle = v_{12} - t v_{22} - t^2 v_{21}.$$
Again using (2.11) and (2.12) we see that this implies that $v_0' = 0$ which is a contradiction.

Finally we prove (4). Since K acts transitively on H/P we see that it is enough to prove (2.10) for $\eta_0 := P$. Since $\omega_{\mathfrak{l}_0}$ is maximal in $\mathcal{H}_{\mathfrak{z}}(\wedge^4 \mathfrak{g})$ by part (1) we deduce that $(\wedge^4 \mathfrak{g})_{\eta_0}[\omega_{\mathfrak{l}_0}] = (\wedge^4 \mathfrak{g})^{\omega_{\mathfrak{l}_0}}$. Thus, fixing $u_0 \in \wedge^3 \mathfrak{r}_0 \smallsetminus \{0\}$, (2.10) becomes
$$\{v \wedge u_0 : v \in \mathfrak{g}, \text{ for all } t \in \mathfrak{z}, \operatorname{ad}_t(v \wedge u_0) = \omega_{\mathfrak{l}_0}(t)(v \wedge u_0)\} = \wedge^4 \mathfrak{l}_0.$$
The inclusion $\supseteq$ is clear as $\omega_{\mathfrak{l}_0}$ was defined to be the weight by which $\mathfrak{z}$ acts on $\wedge^4 \mathfrak{l}_0$. We now establish the inclusion $\subseteq$. Without loss of generality v is orthogonal to $\mathfrak{r}_0$, which means that v is a linear combination of $\{d_2, e_{ij} : (i,j) \notin \{(1,3),(2,3)\}\}$. In turn, $v \wedge u_0$ is a linear combination of $\{d_2 \wedge u_0, e_{ij} \wedge u_0 : (i,j) \notin \{(1,3),(2,3)\}\}$. Since all the vectors in this set are eigenvalues of $\operatorname{ad}_{\mathfrak{z}}$ and only $e_{12} \wedge u_0$ has eigenvalue given by $\omega_{\mathfrak{l}_0}$ we deduce that if $\operatorname{ad}_t(v \wedge u_0) = \omega_{\mathfrak{l}_0}(t)(v \wedge u_0)$ for all $t \in \mathfrak{z}$, then $v \in \mathbb{R}(e_{12} \wedge u_0) = \wedge^4 \mathfrak{l}_0$ which completes the proof. □

Remark 2.8. *Lemma 2.7 lies at the heart of the discussion. In Lemma 2.7 the crucial difference between Case I and Case II manifests itself. Parts* (2) *and* (3) *say that certain vectors in $\wedge^4 \mathfrak{g}$ have a component in $(\wedge^4 \mathfrak{g})[\omega_{\mathfrak{l}_0}]$ which is of 'positive proportion'. This will allow us to use the positivity* (2.7) *and control to some extent the way two nearby points in X drift away from each other.*

Part (4) *is needed to ensure that our 'limiting displacement' will be pointing in the right direction in the case that the multiplicity of $\omega_{\mathfrak{l}_0}$ is larger than 1.*

In §3.5 we will know that the two nearby points lie in the same plane p. Hence, the displacement vector between them corresponds to a pure wedge of the form $v \wedge u$ for $v \in \mathfrak{g}_p$ and $u \in \wedge^3 \mathfrak{r}_p$. This will allow us to use Lemma 2.7 in both Case I and Case II.

On the other hand, in §6 we will use the same positivity to prove Theorem 2.1(b) *which is the statement that the limit measures are non-atomic. There, we will also need to understand how two near-by points in X drift apart but will need to do so for pairs of points which do not necessarily lie in the same plane. This means that the displacement vector between them is of the form $v \wedge u$ where $v \in \mathfrak{g}$ and $u \in \mathfrak{r}_p$ for some $p \in \mathrm{Gr}_2(\mathbb{R}^3)$. Thus, we would be able to apply Lemma 2.7 only for Case I.*

One concludes that the small technical difference between parts (2) *and* (3) *of Lemma 2.7 is what stands behind the phenomenon appearing in Theorem 1.10.*

Remark 2.9. *By analysing the proof of Lemma 2.7 one can see that in Case II the subspace* $\{v \wedge u : v \in \mathfrak{g}, u \in \wedge^3 \mathfrak{r}_0\}$ *does intersect* $\ker \tau_{\omega_{\mathfrak{l}_0}}$ *non-trivially. In fact, this intersection equals* $\{v \wedge u : v \in \operatorname{span}_{\mathbb{R}}(e_{21} + 2e_{32}), u \in \wedge^3 \mathfrak{r}_0\}$*. This should be compared with the construction given in the proof of Theorem 1.10 since* $\Lambda_t = t(e_{21} + 2e_{32}) \operatorname{span}_{\mathbb{Z}}(\{e_1, e_2\})$*.*

The following lemma will be used in §5 and §6 where we will replace μ by μ^{*n_0} for some $n_0 \in \mathbb{N}$ in order to know that the integrals on the left hand sides of equations (2.13) and (2.14) are bounded away from zero uniformly.

Lemma 2.10. *There exist* $\lambda_0 > 0$ *and* $n_0 > 0$ *such that for all* $n \geq n_0$*, the following hold:*
(1) In both Case I and Case II, for all $v \in \mathbb{R}^3 \smallsetminus \{0\}$ *and* $w \in \wedge^2 \mathbb{R}^3 \smallsetminus \{0\}$ *one has*

$$\int_G \log\left(\frac{\|gv\|}{\|v\|} \Big/ \frac{\|gw\|^{1/2}}{\|w\|^{1/2}}\right) \mathrm{d}\mu^{*n}(g) > n\lambda_0. \tag{2.13}$$

(2) In case Case I, for all $p \in \operatorname{Gr}_2(\mathbb{R}^3)$*,* $u \in \wedge^3 \mathfrak{r}_p \smallsetminus \{0\}$ *and* $v \in \mathfrak{g} \smallsetminus \mathfrak{r}_p$ *one has*

$$\int_G \log\left(\frac{\|g(v \wedge u)\|}{\|v \wedge u\|} \Big/ \frac{\|gu\|}{\|u\|}\right) \mathrm{d}\mu^{*n}(g) > n\lambda_0. \tag{2.14}$$

Proof. Let

$$\lambda_1 := \frac{1}{2} \min\{\omega_{\mathbb{R}^3}(\sigma_\mu) - \frac{1}{2}\omega_{\wedge^2\mathbb{R}^3}(\sigma_\mu),\ \omega_{\mathfrak{l}_0}(\sigma_\mu) - \omega_{\mathfrak{r}_0}(\sigma_\mu)\}$$

which is positive by (2.6) and (2.7). The inequality (2.13) with $\lambda_0 = \lambda_1$ and n large enough (independent of the vectors) follows directly from the uniformity of the L^1-convergence in part (3) of Theorem 2.6 applied to the irreducible representations of H on $\mathbb{R}^3$ and $\wedge^2\mathbb{R}^3$.

Next we prove (2.14). First we show that the line $\wedge^3 \mathfrak{r}_p$ in $\wedge^3 \mathfrak{g}$ is contained in the isotypic component $(\wedge^3\mathfrak{g})[\omega_{\mathfrak{r}_0}]$. To see this, note that since $\mathfrak{r}_p = g\mathfrak{r}_0$ where $g \in H$ is such that $gp_0 = p$, it is clear that it is enough to show that the line $\wedge^3\mathfrak{r}_0$ is contained in the isotypic component $(\wedge^3\mathfrak{g})[\omega_{\mathfrak{r}_0}]$. This holds since $\mathfrak{z}$ acts on $\wedge^3\mathfrak{r}_0$ by the weight $\omega_{\mathfrak{r}_0}$ and this line is an eigenline of P.

Next, note that by part (2) of Lemma 2.7 there exists $\alpha > 0$ such that for all $p \in \operatorname{Gr}_2(\mathbb{R}^3)$,$u \in \wedge^3 \mathfrak{r}_p \smallsetminus \{0\}$ and $v \in \mathfrak{g} \smallsetminus \mathfrak{r}_p$ one has $v \wedge u \in (\wedge^4\mathfrak{g})_{\lhd\alpha}[\omega_{\mathfrak{l}_0}]$. This implies that for any $g \in H$ one has

$$\frac{\|g(v \wedge u)\|}{\|v \wedge u\|} \geq \alpha^{-1} \frac{\|g\tau_{\omega_{\mathfrak{l}_0}}(v \wedge u)\|}{\|\tau_{\omega_{\mathfrak{l}_0}}(v \wedge u)\|}.$$

Together with another application of the uniform L^1-convergence in part (3) of Theorem 2.6 (this time for the irreducible representations corresponding to the highest weights $\omega_{\mathfrak{l}_0}$ and $\omega_{\mathfrak{r}_0}$) this shows that (2.14) holds for all large enough n, with $n\lambda_0$ on the right hand side replaced by $n\lambda_1 + \log\alpha$. Since $n\lambda_1 + \log\alpha > n\lambda_1/2$ for all large enough n, the lemma is valid with $\lambda_0 = \lambda_1/2$. □

3. The drift argument - Proof of Theorem 2.1(a)

In this section we will prove Theorem 2.1(a) by adapting the exponential drift argument of Benoist and Quint from [BQ13b]. Throughout $\nu \in \mathcal{P}_\mu(X)$ is a μ-ergodic stationary measure and (B^X, β^X, T) denotes the backwards dynamical system as defined in (2.2).

3.1. **The horocyclic flow.** It will be convenient for us to work in an extension of the backwards dynamical system having an extra coordinate which is used for book keeping purposes. Recall that Z is the Cartan subgroup of H defined in §2.4. Let λ be a Haar measure on Z and let

$$B^{X,Z} := B^X \times Z \quad \text{and} \quad \beta^{X,Z} := \int_{B \times Z} \delta_b \otimes \nu_b \otimes \delta_z \mathrm{d}\beta\,(b)\,\mathrm{d}\lambda\,(z)\,. \tag{3.1}$$

The extension of the backwards dynamical system that we consider is given by the map $\widehat{T} : B^{X,Z} \to B^{X,Z}$ which clearly preserves $\beta^{X,Z}$ and is defined using the Iwasawa cocycle by

$$\widehat{T}(b, x, z) := (Sb, b_1^{-1}x, E(b)^{-1}z).$$

The horocyclic flow is an $\mathbb{R}$-action on $B^{X,Z}$ which interacts with $\widehat{T}$ in a manner reminiscent to the interaction of the standard horocyclic and geodesic flows on the unit tangent bundle of the upper half plane and hence the terminology. Recall the notation introduced in §2.2 and in particular, the groups $L_0, R_0, U_0 := L_0/R_0$ and the resulting equivariant families L_η, R_η, U_η for $\eta \in H/P$ as well as the notation L_b, R_b, U_b defined for β-almost every $b \in B$. We denote the Lie algebras of these groups by corresponding Gothic letters and note that naturally $\mathfrak{u}_0 = \mathfrak{l}_0/\mathfrak{r}_0$ and similar identifications exist when the subscript 0 is replaced by $\eta \in H/P$ or b in the domain of definition of the boundary map ξ. Observe that although for $\eta = gP \in H/P$ the map Ad_g maps $\mathfrak{l}_0$ to $\mathfrak{l}_\eta$ and $\mathfrak{r}_0$ to $\mathfrak{r}_\eta$ and therefore descends to a map $\mathrm{Ad}_g : \mathfrak{u}_0 \to \mathfrak{u}_\eta$, this map is **not** well defined in the sense that it depends on the choice of representative g for the coset η. This is remedied as follows. Recall the section $s : H/P \to H/N$ that was chosen in §2.4 where N is the unipotent radical of the minimal parabolic P of H. Observe that although N acts via the adjoint representation non-trivially on $\mathfrak{l}_0, \mathfrak{r}_0$ respectively, these actions descend to the trivial action on the quotient $\mathfrak{u}_0$. Thus, given $\eta \in H/P$ with $s(\eta) = gN \in H/N$, we do have a well defined map $\mathfrak{u}_0 \to \mathfrak{u}_\eta$ given by

$$\ell + \mathfrak{r}_0 \mapsto \mathrm{Ad}_g(\ell + \mathfrak{r}_0) = \mathrm{Ad}_g(\ell) + \mathfrak{r}_\eta \in \mathfrak{l}_\eta/\mathfrak{r}_\eta = \mathfrak{u}_\eta.$$

By abuse of notation we denote this map

$$\mathrm{Ad}_{s(\eta)} : \mathfrak{u}_0 \to \mathfrak{u}_\eta.$$

Precomposing with the boundary map ξ we obtain the isomorphisms $\mathrm{Ad}_{s(\xi(b))} : \mathfrak{u}_0 \to \mathfrak{u}_b$ defined for β-almost every $b \in B$. Note also that Z acts on $\mathfrak{u}_0$ via the adjoint representation and hence the isomorphism $\mathrm{Ad}_{s(\xi(b))z} = \mathrm{Ad}_{s(\xi(b))}\mathrm{Ad}_z : \mathfrak{u}_0 \to \mathfrak{u}_b$ is well defined for all $z \in Z$ and β-almost every $b \in B$. Following [BQ13b], for $u \in \mathfrak{u}_0$ we define the *horocyclic flow* $\Phi_u : B^{X,Z} \to B^{X,Z}$ by

$$\Phi_u(b, x, z) := (b, \exp(\mathrm{Ad}_{s(\xi(b))z}(u))x, z). \tag{3.2}$$

Further clarification is needed in this definition: For $\eta \in H/P$ and $\bar{\ell} = \ell + \mathfrak{r}_\eta \in \mathfrak{u}_\eta$, $\exp(\bar{\ell}) := \exp(\ell)R_\eta \in U_\eta$ is well defined. Moreover, since the action of R_η on the plane p_η is trivial, the group U_η acts on the fibre $\pi^{-1}(p_\eta) \subset X$. By Proposition 2.2, for β-almost every $b \in B$, ν_b is supported on $\pi^{-1}(p_b)$ and therefore we conclude from (3.1) that for $\beta^{X,Z}$-almost every $(b, x, z) \in B^{X,Z}$ we have that $x \in \pi^{-1}(p_b)$ and $\exp(\mathrm{Ad}_{s(\xi(b))z}(u)) \in U_b$ so equation (3.2) makes sense.

We will utilise the joint action of $\widehat{T}$ and the flow $\Phi_{\mathfrak{u}_0}$ on $B^{X,Z}$. A key point is the following lemma.

Lemma 3.1. *For β-almost every $b \in B$, for any $\mathbf{s} = (b, x, z) \in B^{X,Z}$ one has*

$$\Phi_u \circ \widehat{T}(\mathbf{s}) = \widehat{T} \circ \Phi_u(\mathbf{s}) \quad \textit{for all } u \in \mathfrak{u}_0.$$

Proof. Recall that by (2.5) and the definition of E for β-almost every $b \in B$ one has

$$b_1^{-1} s(\xi(b)) = s(\xi(Sb))\, \mathrm{E}(b)^{-1}. \tag{3.3}$$

For arbitrary $u \in \mathfrak{u}_0$, using the definitions we have that

$$\Phi_u \circ \widehat{T}(b, x, z) = (Sb, \exp(\mathrm{Ad}_{s(\xi(Sb))\,\mathrm{E}(b)^{-1} z}\, u) b_1^{-1} x, \mathrm{E}(b)^{-1} z)$$

and

$$\widehat{T} \circ \Phi_u(b, x, z) = (Sb, b_1^{-1} \exp(\mathrm{Ad}_{s(\xi(b)) z}\, u) x, \mathrm{E}(b)^{-1} z).$$

Once b satisfies (3.3) these two expressions are equal and the lemma follows. □

Later on it will be important for us to restrict attention to a 'finite window' in the Z-coordinate. Let $U \subset Z$ be a bounded measurable set of finite positive λ-measure and define[2]

$$B^{X,U} := B^X \times U \quad \text{and} \quad \beta^{X,U} := \beta^{X,Z}|_{B^{X,U}}. \tag{3.4}$$

Note that $B^{X,U}$ is $\Phi_{\mathfrak{u}_0}$-invariant but not $\widehat{T}$-invariant. The following proposition (in which the role of U is insignificant) shows why the horocyclic flow is natural from the point of view of Proposition 2.4.

Proposition 3.2. *The measure $\beta^{X,U}$ is Φ_u-invariant for all $u \in \mathfrak{u}_0$ if and only if for β-almost every $b \in B$ the measure ν_b is U_b-invariant.*

Remark 3.3. *In particular, by Proposition 2.4, we can prove Theorem 2.1(a) by establishing the $\Phi_{\mathfrak{u}_0}$-invariance of $\beta^{X,U}$.*

Proposition 3.2 is a straightforward corollary of the following lemma.

Lemma 3.4. *Let $\rho : (Y, \eta) \to (Y', \eta')$ be a morphism of Borel probability spaces. Let $\eta = \int_{Y'} \eta_y \mathrm{d}\eta'(y)$ be the disintegration of η over η' and $M : Y \to Y$ be a measurable map such that $\rho = \rho \circ M$. Then η is M-invariant if and only if for η'-almost every $y \in Y'$, η_y is M-invariant.*

The proof of Lemma 3.4 is a direct consequence of the uniqueness of disintegration and is left to the reader.

3.2. **Leafwise measures.** We begin with some general notation and measure theory. Given a locally compact second countable Hausdorff space Y we let $\mathcal{M}(Y)$ denote the space of Radon measures on Y. We equip $\mathcal{M}(Y)$ with the coarsest topology so that $\theta \mapsto \theta(f) := \int_Y f \mathrm{d}\theta$ is continuous for any $f \in C_c(Y)$. We let $\mathbb{P}\mathcal{M}(Y)$ denote the space of equivalence classes of measures in $\mathcal{M}(Y)$ under the equivalence relation of proportionality and equip it with the quotient topology. For $\eta \in \mathcal{M}(Y)$ we denote by $[\eta]$ its equivalence class in $\mathbb{P}\mathcal{M}(Y)$. For $\eta \in \mathcal{M}(Y)$ and a set $V \subset Y$ of finite η-measure, we let $\eta|_V \in \mathcal{P}(V)$ be given by $(\eta|_V)(F) := \eta(F \cap V)/\eta(V)$ for all measurable $F \subseteq V$. Given a countably generated sub-σ-algebra $\mathcal{A}$ of the Borel σ-algebra, the atom of y with respect to $\mathcal{A}$ is the smallest $\mathcal{A}$-measurable set containing y and we denote it by $[y]_{\mathcal{A}}$. Given $\eta \in \mathcal{M}(Y)$, the conditional measures of η along $\mathcal{A}$ are a collection $\{\eta_y^{\mathcal{A}}\}_{y \in E}$ of probability measures $\eta_y^{\mathcal{A}} \in \mathcal{P}(Y)$, where E is a measurable subset of Y of full η-measure such that for any η-integrable function f on Y, the map $y \mapsto \int_Y f \mathrm{d}\eta_y^{\mathcal{A}}$ is the conditional expectation $\mathbb{E}(f|\mathcal{A})$. It then follows that η-almost surely $\eta_y^{\mathcal{A}} \in \mathcal{P}([y]_{\mathcal{A}})$. If Y is a group, then for $y \in Y$ we denote by $l_y : Y \to Y$ the translation by y on the left. This induces an action of Y on $\mathcal{M}(Y)$, $(y, \eta) \mapsto (l_y)_* \eta$. This action respects the equivalence relation of proportionality and hence descends to an action on $\mathbb{P}\mathcal{M}(Y)$ which we denote $(y, [\eta]) \mapsto (l_y)_*[\eta]$.

[2] Later on we will take U to be the image under the exponential map of the unit ball in $\mathfrak{z}$.

We will use the theory of leafwise measures as presented in [EL10, §6], [BQ11, §4]. This is a measure theoretic toolbox developed Katok-Spatzier, Lindenstrauss, Benoist-Quint and Einsiedler-Katok-Lindenstauss [KS98, Lin06, BQ11, EKL06] which captures the way a measure on a space disintegrates with respect to the action of a group. We will follow the notation and terminology of [EL10].

Let $(Y, \mathcal{Y})$ be a standard Borel space and let Ψ_t a measurable $\mathbb{R}$-action[3] (a flow) on Y. Assume that for any $y \in Y$ the stabiliser $\mathrm{Stab}_{\mathbb{R}}(y)$ is a discrete subgroup of $\mathbb{R}$ and let $\eta \in \mathcal{M}(Y)$ be a finite measure. The construction in [BQ11, §4] (see also [EL10, §6]) gives rise to a measurable map $y \mapsto (\eta)_y^{\Psi}$ from a measurable subset of full η-measure $E \subset Y$ to $\mathcal{M}(Y)$ having the following properties:

P1: Characterising property. Given a measurable subset $E' \subset E$ with $\eta(E') > 0$ and a countably generated sub-σ-algebras $\mathcal{A}$ of the Borel σ-algebra on E' whose atoms are of the form $[y]_{\mathcal{A}} = \{\Psi_t(y) : t \in o_y\}$ for some open and bounded $o_y \subset \mathbb{R}$, for all $y \in E'$, then the push-forward of $(\eta)_y^{\Psi}|_{o_y}$ via the orbit map $t \mapsto \Psi_t(y)$ is the conditional measure $(\eta|_{E'})_y^{\mathcal{A}}$.

P2: Rootedness. For all $y \in E$, we have $0 \in \mathrm{supp}\,(\eta)_y^{\Psi}$.

P3: Normalisation. For any $y \in E$, $(\eta)_y^{\Psi}([-1, 1]) = 1$

P4: Compatibility. For all $y \in E$ and $t \in \mathbb{R}$ such that $\Psi_t(y) \in E$ one has

$$[(\eta)_y^{\Psi}] = (l_t)_*[(\eta)_{\Psi_t(y)}^{\Psi}].$$

Property **P1** is a characterising property in the sense that if $y \mapsto \sigma(y)$ is a measurable map defined on a set of full η-measure into $\mathcal{M}(Y)$ such that **P1** is satisfied then $[\sigma(y)] = [(\eta)_y^{\Psi}]$ for η-almost every $y \in Y$. Property **P3** is a convenient way to choose in a measurable manner a well defined measure in the equivalence class $[(\eta)_y^{\Psi}]$ which is well defined for η-almost every $y \in Y$ by **P1, P2**.

We call the map $y \mapsto (\eta)_y^{\Psi}$ satisfying properties **P1-P4** the *leafwise measure-map* (LWM-map) of η with respect to the flow $\Psi_{\mathbb{R}}$ and the set E is called a domain of the LWM-map. The measure $(\eta)_y^{\Psi}$ is called the *leafwise measure* (LWM) of η at y with respect to the flow $\Psi_{\mathbb{R}}$.

We shall consider the LWM-map of the infinite Radon measure $\beta^{X,Z} \in \mathcal{M}(B^{X,Z})$ with respect to the flow $\Phi_{\mathfrak{u}_0}$. The fact that this measure is infinite does not matter much as one can present $B^{X,Z}$ as a countable union of $\Phi_{\mathfrak{u}_0}$-invariant sets of the form $B \times X \times U_i$, where for example U_i is the ball of radius i in Z centred at the identity and the restriction of $\beta^{X,Z}$ to each such set has finite measure. In fact, due to the fact that the flow $\Phi_{\mathfrak{u}_0}$ respects the disintegration $\beta^{X,Z} = \int_{B\times Z} \delta_b \otimes \nu_b \otimes \delta_z \mathrm{d}\beta\,(b)\,\mathrm{d}\lambda\,(z)$ we have the following.

Lemma 3.5. *For $\beta \otimes \lambda$-almost every $(b, z) \in B \times Z$ and $\delta_b \otimes \nu_b \otimes \delta_z$-almost every $\mathbf{s} \in B^{X,Z}$,*

$$(\delta_b \otimes \nu_b \otimes \delta_z)_{\mathbf{s}}^{\Phi} = (\beta^{X,Z})_{\mathbf{s}}^{\Phi}.$$

Proof. Let E be a domain for the LWM-map of $\beta^{X,Z}$ with respect to the flow $\Phi_{\mathfrak{u}_0}$. Let $(b, z) \in B \times Z$ be such that the slice $E_{(b,z)} = \{\mathbf{s} = (b, x, z) \in E\}$ has full $\delta_b \otimes \nu_b \otimes \delta_z$-measure, which holds $\beta \otimes \lambda$-almost surely since E is of full $\beta^{X,Z}$-measure. It is straightforward to check that for β-almost every $b \in B$, the assignment $\mathbf{s} \mapsto (\beta^{X,U})_{\mathbf{s}}^{\Phi}$ satisfies the characterising property of the LWM-map of $\delta_b \otimes \nu_b \otimes \delta_z$ and by the uniqueness of the LWM-map the statement of the lemma follows. □

[3] The theory concerns itself with a locally compact second countable metrizable topological group but we will focus on flows.

The next lemma utilises the commutation relation in Lemma 3.1 and shows that the LWM-map of $\beta^{X,Z}$ is constant along $\widehat{T}$-orbits.

Lemma 3.6. *For $\beta^{X,Z}$-almost every $\mathbf{s} \in B^{X,Z}$ and all $n \in \mathbb{N}$,*

$$(\beta^{X,Z})^{\Phi}_{\mathbf{s}} = (\beta^{X,Z})^{\Phi}_{\widehat{T}^n(\mathbf{s})}. \tag{3.5}$$

Proof. For $\beta \otimes \lambda$-almost every $(b,z) \in B \times Z$,

$$\widehat{T} : (B^{X,Z}, \delta_b \otimes \nu_b \otimes \delta_z) \longrightarrow (B^{X,Z}, \delta_{Sb} \otimes \nu_{Sb} \otimes \delta_{\mathrm{E}(b)^{-1}z})$$

is an isomorphism of probability spaces which by Lemma 3.1 commutes with the flow $\Phi_{\mathfrak{u}_0}$. It thus follows from the uniqueness of the LWM-map that for $\beta \otimes \lambda$-almost every $(b,z) \in B \times Z$ and $\delta_b \otimes \nu_b \otimes \delta_z$-almost every $\mathbf{s} \in B^{X,Z}$ one has the equality $(\delta_b \otimes \nu_b \otimes \delta_z)^{\Phi}_{\mathbf{s}} = (\delta_{Sb} \otimes \nu_{Sb} \otimes \delta_{\mathrm{E}\, b^{-1}z})^{\Phi}_{\widehat{T}(\mathbf{s})}$. Taking into account Lemma 3.5 we deduce that for $\beta^{X,Z}$-almost every $\mathbf{s} \in B^{X,Z}$ the equality $(\beta^{X,Z})^{\Phi}_{\mathbf{s}} = (\beta^{X,Z})^{\Phi}_{\widehat{T}(\mathbf{s})}$ holds. This propagates to the statement of the lemma by intersecting countably many sets of full measure. $\square$

Preparing the grounds for the drift argument we restrict attention to a finite window and consider the probability space $(B^{X,U}, \beta^{X,U})$ as in (3.4). The relevance of the LWM's to our discussion is the following statement. See [EL10, Problem 6.28] and [BQ11, Proposition 4.3].

Theorem 3.7. *The measure $\beta^{X,U}$ is $\Phi_{\mathfrak{u}_0}$-invariant if and only if $(\beta^{X,U})^{\Phi}_{\mathbf{s}}$ is equal to the Haar measure on $\mathfrak{u}_0$ for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$.*

In particular, in order to prove that $\beta^{X,U}$ is $\Phi_{\mathfrak{u}_0}$-invariant we need to show that $(\beta^{X,U})^{\Phi}_{\mathbf{s}}$ is Haar $\beta^{X,U}$-almost surely. Note that by Remark 3.3, this would complete the proof of Theorem 2.1(a). The Haar measure is characterised as the unique $\eta \in \mathcal{M}(\mathfrak{u}_0)$ such that $\mathrm{Stab}_{\mathfrak{u}_0}(\eta) = \mathfrak{u}_0$. Thus our goal is to establish the $\beta^{X,U}$-almost sure equality

$$\mathrm{Stab}_{\mathfrak{u}_0}((\beta^{X,U})^{\Phi}_{\mathbf{s}}) = \mathfrak{u}_0.$$

Because of property **P4** of the LWM's, they interact more naturally with the action of $\mathfrak{u}_0$ on $\mathbb{P}\mathcal{M}(\mathfrak{u}_0)$. In turn, the drift argument in §3.5 will produce the almost sure equality $\mathrm{Stab}_{\mathfrak{u}_0}([(\beta^{X,U})^{\Phi}_{\mathbf{s}}]) = \mathfrak{u}_0$. Hence the importance of the following proposition to our discussion.

Proposition 3.8. *For $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ one has*

$$\mathrm{Stab}_{\mathfrak{u}_0}((\beta^{X,U})^{\Phi}_{\mathbf{s}}) = \mathrm{Stab}_{\mathfrak{u}_0}([(\beta^{X,U})^{\Phi}_{\mathbf{s}}]).$$

Proof. We will prove that

$$\mathrm{Stab}_{\mathfrak{u}_0}((\beta^{X,U})^{\Phi}_{\mathbf{s}}) \supseteq \mathrm{Stab}_{\mathfrak{u}_0}([(\beta^{X,U})^{\Phi}_{\mathbf{s}}])$$

for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ since the reverse inclusion is obvious. By [EL10, Theorem 6.30] for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ we have

$$\lim_{T \to \infty} \frac{1}{T^4} (\beta^{X,U})^{\Phi}_{\mathbf{s}}([-T,T]) = 0. \tag{3.6}$$

Let $\mathbf{s} \in B^{X,U}$ be such a point. If $u \in \mathrm{Stab}_{\mathfrak{u}_0}([(\beta^{X,U})^{\Phi}_{\mathbf{s}}]) \setminus \{0\}$ then there exists $c \neq 0$ such that $(l_u)_*(\beta^{X,U})^{\Phi}_{\mathbf{s}} = c(\beta^{X,U})^{\Phi}_{\mathbf{s}}$, which implies that for all $i \in \mathbb{N}$, $(l_{iu})_*(\beta^{X,U})^{\Phi}_{\mathbf{s}} = c^i(\beta^{X,U})^{\Phi}_{\mathbf{s}}$.

We would like to show that $c = 1$. For all $n \geq 1$ we have

$$\begin{aligned}(\beta^{X,U})^{\Phi}_{\mathbf{s}}([-n|u|, n|u|]) &\geq \sum_{i=1-n}^{n-1} (\beta^{X,U})^{\Phi}_{\mathbf{s}}(l_{iu}[-|u|/2, |u|/2]) \\ &= \sum_{i=1-n}^{n-1} ((l_{iu})_*(\beta^{X,U})^{\Phi}_{\mathbf{s}})([-|u|/2, |u|/2]) \\ &= (\beta^{X,U})^{\Phi}_{\mathbf{s}}([-|u|/2, |u|/2]) \sum_{i=1-n}^{n-1} c^i.\end{aligned}$$

By property **P2** of the LWM's, $(\beta^{X,U})^{\Phi}_{\mathbf{s}}([-|u|/2, |u|/2]) > 0$ and hence we see that unless $c = 1$ the volume $(\beta^{X,U})^{\Phi}_{\mathbf{s}}([-n|u|, n|u|])$ is growing exponentially in n which contradicts (3.6) as desired. □

3.3. **Zooming in on the atoms.** Let $\mathcal{B}^{X,Z}$ be the Borel σ-algebra of $B^{X,Z}$ and define

$$\mathcal{Q}_n := \widehat{T}^{-n}(\mathcal{B}^{X,Z}) \quad \text{and} \quad \mathcal{Q}_\infty := \cap_{i=0}^{\infty} \mathcal{Q}_i.$$

For $\mathbf{s} \in B^{X,Z}$ the atom of $\mathbf{s}$ with respect to $\mathcal{Q}_n$ is given by

$$[\mathbf{s}]_{\mathcal{Q}_n} = \{\mathbf{s}' : \widehat{T}^n(\mathbf{s}) = \widehat{T}^n(\mathbf{s}')\}$$

and the atom of $\mathbf{s}$ with respect to $\mathcal{Q}_\infty$ is given by

$$[\mathbf{s}]_{\mathcal{Q}_\infty} = \{\mathbf{s}' : \text{there exists } n \in \mathbb{N} \text{ with } \widehat{T}^n(\mathbf{s}) = \widehat{T}^n(\mathbf{s}')\}.$$

Recall that $A := \operatorname{supp} \mu$. For $a \in A^n$ we let

$$\mathbf{s}(a) := (aS^n b, a_1^n (b_1^n)^{-1} x, \mathrm{E}_n(aS^n b)\, \mathrm{E}_n(b)^{-1} z). \tag{3.7}$$

We then have an identification $A^n \cong [\mathbf{s}]_{\mathcal{Q}_n}$ via the map $a \mapsto \mathbf{s}(a)$. It is easy to see that via this identification the probability measure $\mu^{\otimes n}$ on A^n corresponds to the conditional measure $(\beta^{X,Z})^{\mathcal{Q}_n}_{\mathbf{s}}$ (cf. [BQ13b, Lemma 3.3]).

We will need to consider σ-algebras whose atoms are tiny parts of the above atoms. This is done as follows. From now on we fix

$$U := \exp(\{z \in \mathfrak{z} : \|z\| < 1\}) \subset Z \tag{3.8}$$

and recall the notation and definition in (3.4). We define $\mathcal{Q}_n^U$ (resp. $\mathcal{Q}_\infty^U$) to be the restriction of $\mathcal{Q}_n$ (resp. $\mathcal{Q}_\infty$) to $B^{X,U}$. For $\mathbf{s} \in B^{X,U}$ the atom of $\mathbf{s}$ with respect to $\mathcal{Q}_n^U$ is given by

$$[\mathbf{s}]_{\mathcal{Q}_n^U} = \{\mathbf{s}(a) : a \in A^n \text{ and } \mathrm{E}_n(aS^n b)\, \mathrm{E}_n(b)^{-1} z \in U\}.$$

We therefore let

$$A^n_{\mathbf{s},U} := A^n_{(b,z),U} := \{a \in A^n : \mathrm{E}_n(aS^n b)\, \mathrm{E}_n(b)^{-1} z \in U\} \tag{3.9}$$

be the subset of A^n corresponding to the subset $[s]_{\mathcal{Q}_n^U}$ of $[s]_{\mathcal{Q}_n}$.

If $\mu^{\otimes n}(A^n_{\mathbf{s},U}) > 0$, then we denote by $\mu^{\otimes n}_{\mathbf{s},U}$ the normalised restriction of $\mu^{\otimes n}$ to $A_{\mathbf{s},U}$. That is

$$\mu^{\otimes n}_{\mathbf{s},U} := \mu^{\otimes n}|_{A_{\mathbf{s},U}}. \tag{3.10}$$

Note that $\mu^{\otimes n}_{\mathbf{s},U}$ only depends on the B and Z co-ordinates of $\mathbf{s}$. By [BQ13b, Lemma 3.6 + Equation (3.5)], under the identification $a \mapsto \mathbf{s}(a)$ of $A^n_{\mathbf{s},U}$ and $[\mathbf{s}]_{\mathcal{Q}_n^U}$ we have that for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$,

$$\mu^{\otimes n}_{\mathbf{s},U} = (\beta^{X,U})^{\mathcal{Q}_n^U}_{\mathbf{s}}. \tag{3.11}$$

We are now ready to cite the essential technical results from [BQ13b] that will allow us to analyse in detail the growth and directions of sequences of vectors corresponding to displacements between points in X. These results are stated in terms of the conditional measures $\mu_{\mathbf{s},U}^{\otimes n}$. For $\delta > 0$ we use the notation $x \asymp_\delta y$ to mean there exists a constant $c_\delta \geq 1$ depending on δ such that that $c_\delta^{-1} x < y < c_\delta x$ for all $x, y \in \mathbb{R}$.

The following lemma is used in order to control the growth of displacements.

Lemma 3.9. *Let V be a finite dimensional representation of H. For $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ and all $\delta > 0$ there exists $n_0 > 0$ such that for all $n > n_0$, $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ and $v \in V[\omega] \smallsetminus \{0\}$ one has*

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \|a_1^n v\| \asymp_\delta \|a_1^n\|\|v\|\}) > 1 - \delta. \tag{3.12}$$

Proof. This is the first part of [BQ13b, Proposition 4.21] where the conditional measures $\beta_{n,c}^U$ (in the notation of Benoist and Quint) equal $\mu_{\mathbf{s},U}^{\otimes n}$ [BQ13b, Lemma 3.6 + Equation (3.5)]. □

We will use Lemma 3.9 in the following form.

Corollary 3.10. *Let V be a finite dimensional representation of H. Then for $\beta^{X,U}$-almost every $\mathbf{s} = (b, x, z) \in B^{X,U}$ and all $\delta > 0$ there exists $n_0 > 0$ such that for all $n > n_0$, $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ and $v \in V[\omega] \smallsetminus \{0\}$ one has*

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \|a_1^n v\| \asymp_\delta \chi^\omega(\mathrm{E}_n(b))\|v\|\}) > 1 - \delta. \tag{3.13}$$

Proof. Let $\mathbf{s} = (b, x, z) \in B^{X,U}$ be such that the conclusion of Lemma 3.9 holds for $\mathbf{s}$. Given $\delta > 0$ we get the existence of n_0 such that (3.12) holds for all $n > n_0$, $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ and all $v \in V[\omega] \smallsetminus \{0\}$.

Let $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ and $v_0 \in V_{S^n b}[\omega]$ a unit vector. By Lemma 2.5, if $\mathbf{s}$ is outside a $\beta^{X,U}$-null set, for $\mu^{\otimes n}$-almost every $a \in A^n$ one has

$$\|a_1^n v_0\| / \|v_0\| = \chi^\omega(\mathrm{E}_n(a S^n b)).$$

Applying equation (3.12) to v_0 we get that for all $n > n_0$ and $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$,

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \|a_1^n\| \asymp_\delta \chi^\omega(\mathrm{E}_n(a S^n b))\}) > 1 - \delta. \tag{3.14}$$

Taking into account that we are conditioning on the fact that

$$\mathrm{E}_n(a S^n b)\, \mathrm{E}_n(b)^{-1} z \in U$$

and that $z \in U$, we may replace $\mathrm{E}_n(a S^n b)$ with $\mathrm{E}_n(b)$ in (3.14) by modifying the implied constant if necessary. This gives us that for all $n > n_0$ and $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$,

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \|a_1^n\| \asymp_\delta \chi^\omega(\mathrm{E}_n(b))\}) > 1 - \delta.$$

Applying again (3.12) we get that for all $n > n_0$, $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ and $v \in V[\omega] \smallsetminus \{0\}$ one has

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \|a_1^n v\| \asymp_\delta \chi^\omega(\mathrm{E}_n(b))\|v\|\}) > 1 - 2\delta,$$

which finishes the proof up to replacing δ by $\delta/2$. □

The following lemma will allow us to control the direction of displacements. For any vector space V we use the distance $\mathrm{d}_{\mathbb{P}V}$ on $\mathbb{P}V$ defined so that for all $v \in V \smallsetminus \{0\}$ and $W \subseteq V$ one has

$$\mathrm{d}_{\mathbb{P}V}(\mathbb{R}v, W) := \min_{w \in W} \frac{\|v \wedge w\|}{\|v\|\|w\|}.$$

Note that $\mathrm{d}_{\mathbb{P}V}(\mathbb{R}v, W) = 0$ if and only if $\mathbb{R}v \subseteq W$.

Lemma 3.11. *Let V be a finite dimensional representation of H. Then for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ and for all $\rho > 0$ and $\delta > 0$ there exists $n_0 > 0$ such that for all $n > n_0$, $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$, $v \in V[\omega] \smallsetminus \{0\}$ and $\eta \in H/P$ one has*

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \mathrm{d}_{\mathbb{P}V}(a_1^n \mathbb{R}v, a_1^n V_\eta[\omega]) < \rho\}) > 1 - \delta. \tag{3.15}$$

Proof. This is exactly the second part of [BQ13b, Proposition 4.21] where the conditional measures $\beta_{n,c}^U$ (in the notation of Benoist and Quint) equal $\mu_{\mathbf{s},U}^{\otimes n}$ by [BQ13b, Lemma 3.4 + Equation (3.5)]. □

In our application of Lemma 3.11 we will not know that the vector v belongs to a single isotypic component. The following lemma will allow us to obtain similar statements for vectors which do not lie in a single isotypic component.

Lemma 3.12. *Let V be a representation of H and assume that $\mathcal{H}_{\mathfrak{z}}(V)$ contains a maximal weight $\omega_{\mathbf{m}}$. Let $\alpha > 0$ and $V_{\lhd\alpha}[\omega_{\mathbf{m}}]$ be as in (2.9). Then for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ and for all $\rho > 0$ and $\delta > 0$ there exists $n_0 > 0$ such that for all $n > n_0$ and $v \in V_{\lhd\alpha}[\omega_{\mathbf{m}}] \smallsetminus \{0\}$ one has*

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \mathrm{d}_{\mathbb{P}V}(a_1^n \mathbb{R}v, a_1^n \mathbb{R}\tau_{\omega_{\mathbf{m}}}(v)) \le \rho\}) > 1 - \delta. \tag{3.16}$$

Proof. Let $v \in V_{\lhd\alpha}[\omega_{\mathbf{m}}] \smallsetminus \{0\}$. Then for any $g \in H$ one has

$$\begin{aligned} \mathrm{d}_{\mathbb{P}V}(g\mathbb{R}v, g\mathbb{R}\tau_{\omega_{\mathbf{m}}}(v)) &= \frac{\|(\sum_{\omega \in \mathcal{H}_{\mathfrak{z}}(V)} g\tau_\omega(v)) \wedge g\tau_{\omega_{\mathbf{m}}}(v)\|}{\|gv\|\|g\tau_{\omega_{\mathbf{m}}}(v)\|} \\ &= \frac{\|\sum_{\omega \in \mathcal{H}_{\mathfrak{z}}(V) \smallsetminus \{\omega_{\mathbf{m}}\}} g\tau_\omega(v)\|\|g\tau_{\omega_{\mathbf{m}}}(v)\|}{\|gv\|\|g\tau_{\omega_{\mathbf{m}}}(v)\|} \\ &\ll \frac{\max_{\omega \in \mathcal{H}_{\mathfrak{z}}(V) \smallsetminus \{\omega_{\mathbf{m}}\}} \|g\tau_\omega(v)\|}{\|g\tau_{\omega_{\mathbf{m}}}(v)\|}. \end{aligned} \tag{3.17}$$

Now given $\rho > 0$, by parts (2) and (4) of Theorem 2.6, for β-almost every $b \in B$ there exists $n_0 > 0$ so that for all $n > n_0$ and $\omega \in \mathcal{H}_{\mathfrak{z}}(V) \smallsetminus \{\omega_{\mathbf{m}}\}$ one has

$$\exp((\omega - \omega_{\mathbf{m}})(\mathrm{L}_n(b))) = \frac{\chi^\omega(\mathrm{E}_n(b))}{\chi^{\omega_{\mathbf{m}}}(\mathrm{E}_n(b))} \le \rho. \tag{3.18}$$

By enlarging n_0 if necessary and using Corollary 3.10 we get that for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ and for all $\delta > 0$ and $n > n_0$ there is $F \subset A^n$ with $\mu_{\mathbf{s},U}^{\otimes n}(F) > 1 - \delta$ such that for all $a \in F$, $\omega \in \mathcal{H}_{\mathfrak{z}}(V)$ and $v \in V$ such that $\tau_\omega(v) \neq 0$ we have $\|a_1^n \tau_\omega(v)\| \asymp_\delta \chi^\omega(\mathrm{E}_n(b))\|\tau_\omega(v)\|$. Thus, using (3.17), (3.18) and the assumption that $v \in V_{\lhd\alpha}[\omega_{\mathbf{m}}] \smallsetminus \{0\}$ we get

$$\mathrm{d}_{\mathbb{P}V}(a_1^n \mathbb{R}v, a_1^n \mathbb{R}\tau_{\omega_{\mathbf{m}}}(v)) \ll_\delta \rho/\alpha$$

for all $a \in F$ which, up to adjusting ρ, is the claim of the lemma. □

The following lemma will allow us to upgrade measurability to continuity on certain compact sets of arbitrarily large measure. During the course of the proof and in §3.5 we will use a few standard results from measure theory and analysis such as Lusin's theorem and the martingale convergence theorem. A suitable reference for all of these results is [Bog07].

Lemma 3.13. *Let $E \subset B^{X,U}$ be a measurable subset such that $\beta^{X,U}(E) = 1$. Then, for any $0 < \delta < 1$ there exist compact subsets $K' \subset K \subset E$ such that:*

(1) The map $\mathbf{s} \mapsto (\beta^{X,U})_{\mathbf{s}}^\Phi$ is defined and continuous on K.
(2) The map $\mathbf{s} = (b, x, z) \mapsto \xi(b)$ is defined and continuous on K (see §2.2).
(3) The volume $\beta^{X,U}(K') > 1 - 2\delta$.

(4) There exists $n_0 > 0$ *such that for all* $\mathbf{s} \in K'$ *and* $n > n_0$ *one has*

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \mathbf{s}(a) \in K\}) > 1 - \delta. \tag{3.19}$$

Proof. Let $E \subset B^{X,U}$ be a set of full $B^{X,U}$-measure and $0 < \delta < 1$ be given. We may assume that E is contained in the domain of the LWM-map and the projection to B of E is contained in the full measure set on which ξ is defined and measurable. Hence by Lusin's theorem we may pick a compact set $K \subset E$ such that requirements (1) and (2) hold and such that $\beta^{X,U}(K) > 1 - \delta^2$. Since $0 \leq \mathbb{E}(\mathbf{1}_K|\mathcal{Q}_\infty^U) \leq 1$ and

$$\int_{B^{X,U}} \mathbb{E}(\mathbf{1}_K|\mathcal{Q}_\infty^U)\mathrm{d}\beta^{X,U} > 1 - \delta^2,$$

by Chebyshev's inequality there exists a compact $L' \subset B^{X,U}$ such that $\varphi|_{L'} > 1 - \delta$ and $\beta^{X,U}(L') > 1 - \delta$. Let $L = L' \cap K$ so that $\beta^{X,U}(L) > 1 - \delta - \delta^2 > 1 - 2\delta$.

Since the conditional expectations $\mathbb{E}(\mathbf{1}_K|\mathcal{Q}_n^U)$ are a reversed martingale, by the martingale convergence theorem we have

$$\lim_{n\to\infty} \mathbb{E}(\mathbf{1}_K|\mathcal{Q}_n^U) = \mathbb{E}(\mathbf{1}_K|\mathcal{Q}_\infty^U) \quad \beta^{X,U}\text{-almost surely.}$$

Using Egoroff's theorem we can assume that on L the convergence is uniform. In particular, there exists $n_0 > 0$ such that for all $n > n_0$ one has $\varphi_n|_L > 1 - \delta$. From (3.11) we see that

$$\mathbb{E}(\mathbf{1}_K|\mathcal{Q}_n^U)(\mathbf{s}) = \int_{A^n} \mathbf{1}_K(\mathbf{s}(a))\mathrm{d}\mu_{\mathbf{s},U}^{\otimes n}(a)\,.$$

Hence, for all $\mathbf{s} \in L$ and $n > n_0$ one has

$$\mu_{\mathbf{s},U}^{\otimes n}(\{a \in A^n : \mathbf{s}(a) \in K\}) > 1 - \delta. \tag{3.20}$$

Hence the requirements of the lemma are satisfied with the sets $L \subset K \subset E$. □

3.4. **Constructing the displacements.** We set up some notational conventions which will be used in the drift argument in the next subsection. For $\eta \in H/P$ we consider the quotient $\mathfrak{g}_\eta/\mathfrak{r}_\eta$ of the Lie algebra $\mathfrak{g}_\eta$ of G_η by the Lie algebra $\mathfrak{r}_\eta$ of the solvable radical $R_\eta < G_\eta$. The exponential map $\exp : \mathfrak{g}_\eta/\mathfrak{r}_\eta \to G_\eta/R_\eta$ is well defined and since R_η acts trivially on the plane p_η, the quotient G_η/R_η acts (transitively) on the fibre $\pi^{-1}(p_\eta)$. In particular it makes sense to write for $v \in \mathfrak{g}_\eta/\mathfrak{r}_\eta$ and $x \in \pi^{-1}(p_\eta)$, $\exp(v)x$ and in fact, on letting v vary in a basis of neighbourhoods of 0 in $\mathfrak{g}_\eta/\mathfrak{r}_\eta$ one obtains a basis of neighbourhoods of x in $\pi^{-1}(p_\eta)$. If $y = \exp(v)x$ we refer to v as a *displacement* between y and x.

For $g \in G$, the adjoint action of g on $\mathfrak{g}$ induces an isomorphism from $\mathfrak{g}_\eta/\mathfrak{r}_\eta$ to $\mathfrak{g}_{g\eta}/\mathfrak{r}_{g\eta}$. Thus, for $v \in \mathfrak{g}_\eta/\mathfrak{r}_\eta$ we let gv denote the corresponding image in $\mathfrak{r}_{g\eta}/\mathfrak{r}_{g\eta}$. If $x, y \in \pi^{-1}(p_\eta)$ and $v \in \mathfrak{g}_\eta/\mathfrak{r}_\eta$ is a displacement between x and y, then for any $g \in G$ we have that $gv \in \mathfrak{g}_{g\eta}/\mathfrak{r}_{g\eta}$ is a displacement between $gx, gy \in \pi^{-1}(p_{g\eta})$. In particular, for β-almost every $b \in B$ (where ξ is defined and equivariant) and for all $x, y \in \pi^{-1}(p_b)$, $v \in \mathfrak{g}_b/\mathfrak{r}_b$ and $n \in \mathbb{N}$ one has that:

$$\text{if } \exp(v)x = y \text{ then } \exp((b_1^n)^{-1}v)(b_1^n)^{-1}x = (b_1^n)^{-1}y. \tag{3.21}$$

Remark 3.14. *Note that as* $\mathfrak{r}_\eta$ *is not an ideal in* $\mathfrak{g}$ *these notions cannot be extended to define displacements in* $\mathfrak{g}/\mathfrak{r}_\eta$ *between nearby points* $x, y \in X$ *without the assumption that they both lie in the same plane. In §6 we will need this more general notion of displacement and develop the necessary notation and terminology.*

We equip $\mathfrak{g}_\eta/\mathfrak{r}_\eta$ with the quotient norm which is induced by our pre-fixed inner product on $\mathfrak{g}$. We choose a metric d_X on X in such a way that if $v \in \mathfrak{g}_\eta/\mathfrak{r}_\eta$ and $\|v\| \le \epsilon$, then for any $x \in \pi^{-1}(p_\eta)$ one has that $\mathrm{d}_X(x, \exp(v)x) \le \epsilon$. See §6.1 for details regarding an explicit choice of such a metric.

We use the assumption that the ν_b's are non-atomic β-almost surely to build a sequence of displacements that will become input for the drift argument as reflected in the following lemma.

Lemma 3.15. *Let $F \subset B^{X,Z}$ be a set of positive $\beta^{X,Z}$-measure and suppose that for β-almost every $b \in B$ the measures ν_b are non-atomic. Then, for $\beta^{X,Z}$-almost every $(b,x,z) \in F$ there exists a sequence $\{v_i\}_{i\in\mathbb{N}} \subset \mathfrak{g}_b/\mathfrak{r}_b \smallsetminus \{0\}$ tending to 0 such that that for all $i \in \mathbb{N}$ one has $(b, \exp(v_i)x, z) \in F$ and $\limsup_{n\to\infty} t_n(b, v_i) = \infty$ where*

$$t_n(b, v_i) := \chi_{\mathfrak{l}_0/\mathfrak{r}_0}(\mathrm{E}_n(b))\|(b_1^n)^{-1}v\|. \tag{3.22}$$

Proof. We fix a measurable set $F \subset B^{X,Z}$ such that $\beta^{X,Z}(F) > 0$. Since the statement we are trying to prove is an almost sure statement, it is safe to neglect $\beta^{X,Z}$-null sets. It follows from Proposition 2.2 that we may assume $\operatorname{supp}\nu_b \subseteq \pi^{-1}(p_b)$ and $x \in \pi^{-1}(p_b)$ for all $(b,x,z) \in F$. Furthermore, using the definition of $\beta^{X,Z}$, we may assume that for all $(b,x,z) \in F$, x belongs to the support of ν_b. In other words, if for $i \in \mathbb{N}$ we let $\mathcal{N}_i^b$ denote a basis of neighbourhoods of 0 in $\mathfrak{g}_b/\mathfrak{r}_b$ then for all $(b,x,z) \in F$ and $i \in \mathbb{N}$,

$$\nu_b(\{\exp(v)x : v \in \mathcal{N}_i^b\}) > 0. \tag{3.23}$$

For $b \in B$ let

$$\mathfrak{s}_b := \{v \in \mathfrak{g}_b/\mathfrak{r}_b : \limsup_{n\to\infty} t_n(b,v) < \infty\}.$$

In light of (3.23) and the definition of the measure $\beta^{X,Z}$ in order to prove the lemma, it is enough to establish

$$\nu_b(\exp(\mathfrak{s}_b)x) = 0 \text{ for } \beta^X\text{-almost every } (b,x) \in B^X. \tag{3.24}$$

Let d_X denote a distance function on X as discussed before the lemma. For $(b,x) \in B^X$ let

$$W_b(x) := \{y \in X : \lim_{n\to\infty} \mathrm{d}_X((b_1^n)^{-1}y, (b_1^n)^{-1}x) \to 0\}.$$

It is shown in [BQ13b, Proposition 6.18] that β^X-almost surely $\nu_b(W_b(x) \smallsetminus \{x\}) = 0$. Due to our non-atomicity assumption we deduce that β^X-almost surely $\nu_b(W_b(x)) = 0$. Hence we can verify (3.24) by showing that

$$\exp(\mathfrak{s}_b)x \subset W_b(x) \text{ for } \beta^X\text{-almost every } (b,x) \in B^X. \tag{3.25}$$

To this end, let $(b,x) \in B^X$ and $v \in \mathfrak{s}_b$ so that $t_n(b,v)$ is bounded and let $y = \exp(v)x$. We will finish by showing that if (b,x) is outside a β^X-null set, then $y \in W_b(x)$. By part (2) of Theorem 2.6 and (2.7) one has

$$\lim_{n\to\infty} \omega_{\mathfrak{l}_0/\mathfrak{r}_0}(\mathrm{L}_n(b)/n) = \omega_{\mathfrak{l}_0/\mathfrak{r}_0}(\sigma_\mu) > 0 \quad \beta\text{-almost surely}$$

and hence

$$\limsup_{n\to\infty} \chi_{\mathfrak{l}_0/\mathfrak{r}_0}(\mathrm{E}_n(b)) = \infty \quad \beta\text{-almost surely}. \tag{3.26}$$

Therefore, once (b,x) is such that (3.26) holds then taking into account the definition of $t_n(b,v)$ and its boundedness we conclude that $\lim_{n\to\infty}\|(b_1^n)^{-1}v\| = 0$. In particular, on denoting $x_n = (b_1^n)^{-1}x$ we get that

$$\begin{aligned}\mathrm{d}_X((b_1^n)^{-1}x, (b_1^n)^{-1}y) &= \mathrm{d}_X(x_n, (b_1^n)^{-1}\exp(v)x) \\ &= \mathrm{d}_X(x_n, \exp((b_1^n)^{-1}v)x_n).\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \mathrm{d}_X((b_1^n)^{-1}x, (b_1^n)^{-1}y) = 0$$

where we use (3.21) which holds β-almost surely. This shows that $y \in W_b(x)$ and finishes the proof of the lemma. $\square$

3.5. **The exponential drift - Proof of Theorem 2.1(a).** We now prove Theorem 2.1(a) which we restate for convenience.

Theorem 3.16. *Let $\mu \in \mathcal{P}(G)$ be a compactly supported measure and suppose we are either in Case I or Case II. Let $\nu \in \mathcal{P}_\mu(X)$ be an ergodic μ-stationary measure on X and assume that for β-almost every $b \in B$ the limit measures ν_b are non-atomic, then ν is the natural lift of the Furstenberg measure of μ on $\mathrm{Gr}_2(\mathbb{R}^3)$.*

Proof. Let U be as in (3.8). By Proposition 2.4 and Proposition 3.2 it is enough to establish that $\beta^{X,U}$ is invariant under the horocyclic flow $\Phi_{\mathfrak{u}_0}$. By Theorem 3.7 we are reduced to establishing that for $\beta^{X,U}$-almost every $\mathbf{s} \in B^{X,U}$ the LWM, $(\beta^{X,U})^\Phi_{\mathbf{s}}$ is equal to the Haar measure on $\mathfrak{u}_0$. Said differently, we are reduced to establishing the equality $\mathrm{Stab}_{\mathfrak{u}_0}((\beta^{X,U})^\Phi_{\mathbf{s}}) = \mathfrak{u}_0$, $\beta^{X,U}$-almost surely. By Proposition 3.8 it is enough to establish the following claim:

Claim 3.17. *The equality $\mathrm{Stab}_{\mathfrak{u}_0}([(\beta^{X,U})^\Phi_{\mathbf{s}}]) = \mathfrak{u}_0$ holds $\beta^{X,U}$-almost surely.*

The rest of the proof is devoted to proving this claim. There exists a measurable S-invariant set of full measure $B_0 \subset B$ such that for all $b \in B_0$, the boundary map ξ is defined and equivariant at b and Lemma 2.5 is applicable to b with respect to the exterior powers of the adjoint representation of H on $\mathfrak{g}$.

Let $E \subseteq B_0 \times X \times U$ be a measurable subset of full $\beta^{X,U}$-measure such that the LWM-map is defined on E, Lemma 3.6 is applicable for any point in E in the sense that for all $\mathbf{s} \in E$ and $n \in \mathbb{N}$,

$$(\beta^{X,U})^\Phi_{\mathbf{s}} = (\beta^{X,U})^\Phi_{\widehat{T}^n(\mathbf{s})}. \tag{3.27}$$

Additionally, using Proposition 2.2, we assume that for all $(b, x, z) \in E$ one has $\nu_b(\pi^{-1}(p_b)) = 1$ and $x \in \pi^{-1}(p_b)$. For $\mathbf{s} = (b, x, z) \in E$ and $v \in \mathfrak{g}_b/\mathfrak{r}_b$ we denote

$$\exp(v)\mathbf{s} := (b, \exp(v)x, z). \tag{3.28}$$

Let $0 < \delta < 1/10$ be arbitrarily small and let $K' \subset K \subset E$ be compact subsets be as guaranteed by Lemma 3.13.

Definition. *Given a point $\mathbf{s} = (b, x, z) \in K'$ we say that a sequence $\{v_i\}_{i\in\mathbb{N}} \subset \mathfrak{g}_b/\mathfrak{r}_b$ of non-zero vectors converging to 0 is unstable for $\mathbf{s}$ if*

$$\mathbf{s}_i := \exp(v_i)\mathbf{s} \in K' \quad \textit{for all } i \in \mathbb{N} \tag{3.29}$$

and for any fixed $i \in \mathbb{N}$ the sequence

$$t_n(b, v_i) := \chi_{\mathfrak{l}_0/\mathfrak{r}_0}(\mathrm{E}_n(b))\|(b_1^n)^{-1}v_i\| \tag{3.30}$$

in the variable n is unbounded. Although we do not record in this terminology the set K', it should cause no confusion because K' will remain fixed until the last step of the proof.

By Lemma 3.15, $\beta^{X,U}$-almost every $\mathbf{s} \in K'$ has an unstable sequence. We note that this is the part of the proof where the non-atomicity of the ν_b's is being used.

Let $\mathbf{s} \in K'$ and $\{v_i\}_{i\in\mathbb{N}}$ be an unstable sequence for $\mathbf{s}$. For all i, n and $a \in A^n$ such that $aS^n b \in B_0$ the relation (3.29) between $\mathbf{s}$ and $\mathbf{s}_i$ propagates to a similar relation between $\mathbf{s}(a)$ and $\mathbf{s}_i(a)$. Namely, using the notations from (3.7) and (3.28)

$$\mathbf{s}_i(a) = \exp(a_1^n (b_1^n)^{-1} v_i)\mathbf{s}(a). \tag{3.31}$$

The assumption that $aS^n b \in B_0$ is used in order for (3.21) to apply.

The proof of Claim 3.17 relies on showing that for arbitrarily large $i \in \mathbb{N}$ one can choose carefully $n_i \in \mathbb{N}$ and $a_i \in A^{n_i}$ in such a way that equation (3.31) limits to an equation giving rise to the fact that $[(\beta^{X,U})_{\mathbf{s}}^{\Phi}]$ is invariant under an arbitrarily small element of $\mathfrak{u}_0$. It is quite long and so we try to break it into steps and introduce auxiliary notation and terminology to ease the complications.

Definition. *We say that a point* $\mathbf{s} \in K'$ *satisfies hypothesis* ED *if there exists an unstable sequence* $\{v_i\}_{i\in\mathbb{N}}$ *for* $\mathbf{s}$ *such that for all* $\epsilon > 0$*, for all* $i \in \mathbb{N}$ *there exists choices* $n_i \in \mathbb{N}$ *and* $a_i \in A^{n_i}_{\mathbf{s},U}$ *such that* $n_i \to \infty$ *as* $i \to \infty$ *and the following hold:*

ED1: *For all* $i \in \mathbb{N}$ *one has that the points* $\mathbf{s}(a_i), \mathbf{s}_i(a_i) \in K$*.*

ED2: $\|(a_i)_1^{n_i}(b_1^{n_i})^{-1} v_i\| \asymp \epsilon$.

ED3: $\lim_{i\to\infty} \mathrm{d}_{\mathbb{P}\wedge^4\mathfrak{g}}((a_i)_1^{n_i}(b_1^{n_i})^{-1}(\mathbb{R}\tilde{v}_i \wedge (\wedge^3 \mathfrak{r}_b)), (\wedge^4\mathfrak{g})_{a_i S^{n_i} b}[\omega_{\mathfrak{l}_0}]) = 0$*, where* $\tilde{v}_i \in \mathfrak{g}_b$ *is a representative of* v_i*.*

We note that property **ED1** implies that the B-coordinate of the points $\mathbf{s}(a_i)$ and $\mathbf{s}_i(a_i)$, which is $a_i S^{n_i} b$, belongs to B_0 by our assumption on E. As explained above, this implies that (3.31) holds and moreover, from the definition of B_0 we have the equality

$$(a_i)_1^{n_i}(b_1^{n_i})^{-1}\xi_b = \xi_{a_i S^{n_i} b}. \tag{3.32}$$

We complete the proof of Claim 3.17 in two steps by proving:

(Step 1) $\beta^{X,U}$-almost every $\mathbf{s} \in K'$ satisfies hypothesis ED.

(Step 2) If $\mathbf{s} \in K'$ satisfies hypothesis ED then $\mathrm{Stab}_{\mathfrak{u}_0}[(\beta^{X,U})_{\mathbf{s}}^{\Phi}] = \mathfrak{u}_0$.

Indeed, by part (3) of Lemma 3.13, $\beta^{X,U}(K') \geq 1 - 2\delta$, and since δ is arbitrary the claim follows.

Proof of Step 1. As mentioned before, Lemma 3.15 implies that for $\beta^{X,U}$-almost every $\mathbf{s} \in K'$ there exists an unstable sequence $\{v_i\}_{i\in\mathbb{N}}$. Therefore, there is no problem fixing $\mathbf{s} \in K'$ and $\{v_i\}_{i\in\mathbb{N}}$ a corresponding unstable sequence. Let $\epsilon > 0$.

Fix $i \in \mathbb{N}$ and consider the sequence $t_n = t_n(b, v_i)$ from (3.30). Note that since the support of μ is compact, the ratios t_{n+1}/t_n are bounded by a constant depending on μ. By the definition of the instability of $\{v_i\}_{i\in\mathbb{N}}$ for $\mathbf{s}$, the sequence t_n is unbounded and since t_1 is arbitrarily small for all large i, we conclude that for all large i the number $n_i := \min\{n : t_n > \epsilon\}$ is well defined and in that case

$$t_{n_i} \asymp \epsilon. \tag{3.33}$$

Note that since $v_i \to 0$, we must have that $n_i \to \infty$ as $i \to \infty$. The existence of $a_i \in A^{n_i}$ for which properties **ED1**-**ED3** will hold will be established by probabilistic means using the conditional probability measure $\mu^{\otimes n_i}_{\mathbf{s},U}$ discussed in §3.3.

First we demonstrate that property **ED1** holds for a set of large $\mu^{\otimes n_i}_{\mathbf{s},U}$-measure. Since $n_i \to \infty$ and both of $\mathbf{s}$ and $\mathbf{s}_i$ are elements of K', by Lemma 3.13, which we used to obtain K' and K, we have that: For $\beta^{X,U}$-almost every $\mathbf{s} \in K'$,

$$\mu^{\otimes n_i}_{\mathbf{s},U}(\{a \in A^{n_i} : \mathbf{s}(a), \mathbf{s}_i(a) \in K\}) > 1 - 2\delta \quad \text{for all } i \gg 1. \tag{$\star$}$$

We turn to property **ED2**. Observe that for $c \in B_0$ and $v \in \mathfrak{g}_c/\mathfrak{r}_c$,

$$\|v\| = \frac{\|\tilde{v} \wedge u_c\|}{\|u_c\|} \tag{3.34}$$

where we will use $\tilde{v} \in \mathfrak{g}_c$ to denote a choice of a representative for v and u_c to denote a non-zero element of $\wedge^3 \mathfrak{r}_c$ (note that the quantity in (3.34) does not depend on our choices). This will allow us to obtain **ED2** by considering the representations of H on $\wedge^4 \mathfrak{g}$ and $\wedge^3 \mathfrak{g}$.

For $i \in \mathbb{N}$ we use the notation

$$\mathbf{v}_i := (b_1^{n_i})^{-1}(\tilde{v}_i \wedge u_b) \quad \text{and} \quad \mathbf{v}'_i := (b_1^{n_i})^{-1} u_b. \tag{3.35}$$

For $a \in A^{n_i}$ our goal is to understand the norm in **ED2**. We will compare the quantities

$$\|a_1^{n_i}(b_1^{n_i})^{-1} v_i\| = \frac{\|a_1^{n_i}\mathbf{v}_i\|}{\|a_1^{n_i}\mathbf{v}'_i\|} \quad \text{and} \quad t_{n_i}(b, v_i) = \frac{\chi_{\mathfrak{l}_0}(\mathrm{E}_n(b))\|\mathbf{v}_i\|}{\chi_{\mathfrak{r}_0}(\mathrm{E}_n(b))\|\mathbf{v}'_i\|} \tag{3.36}$$

and show that they are of the same order of magnitude. We start by relating the numerators of the ratios in (3.36) and then consider the corresponding denominators.

We apply Corollary 3.10 to the representation of H on $V = \wedge^4 \mathfrak{g}$ and for the weight $\omega_{\mathfrak{l}_0} \in \mathcal{H}_{\mathfrak{z}}(V)$ and the vector $\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i) \in V[\omega_{\mathfrak{l}_0}]$ and conclude that for $\beta^{X,U}$-almost every $\mathbf{s} \in K'$,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : \|a_1^{n_i}\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i)\| \asymp \chi_{\mathfrak{l}_0}(\mathrm{E}_{n_i}(b))\|\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i)\|\}) > 1 - \delta \quad \text{for all } i \gg 1. \tag{3.37}$$

We wish to replace in (3.37) the term $\|\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i)\|$ by $\|\mathbf{v}_i\|$ and $\|a_1^{n_i}\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i)\| = \|\tau_{\omega_{\mathfrak{l}_0}}(a_1^{n_i}\mathbf{v}_i)\|$ by $\|a_1^{n_i}\mathbf{v}_i\|$. For this we use parts (2) and (3) of Lemma 2.7. In order for Lemma 2.7 to be applicable we need that $\mathbf{v}_i \in \mathfrak{g}_{S^{n_i}b} \wedge (\wedge^3 \mathfrak{r}_{S^{n_i}b})$ and $a_1^{n_i}\mathbf{v}_i \in \mathfrak{g}_{aS^{n_i}b} \wedge (\wedge^3 \mathfrak{r}_{aS^{n_i}b})$. The first containment holds since $b \in B_0$ and the relevant spaces vary equivariantly. For the second containment, if we require a to be also in the set measured in ($\star$) then $aS^{n_i}b \in B_0$ as well and the relevant equivariance applies. This leads us to conclude from ($\star$) and (3.37) that for $\beta^{X,U}$-almost every $\mathbf{s} \in K'$,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : \|a_1^{n_i}\mathbf{v}_i\| \asymp \chi_{\mathfrak{l}_0}(\mathrm{E}_{n_i}(b))\|\mathbf{v}_i\|\}) > 1 - 3\delta \quad \text{for all } \gg 1. \tag{3.38}$$

Regarding the denominators in (3.36), we claim that for a's which are measured in ($\star$),

$$\|a_1^{n_i}\mathbf{v}'_i\| = \chi_{\mathfrak{r}_0}(\mathrm{E}_{n_i}(aS^{n_i}b))\|\mathbf{v}'_i\| \asymp \chi_{\mathfrak{r}_0}(\mathrm{E}_{n_i}(b))\|\mathbf{v}'_i\| \quad \text{for all } i \gg 1. \tag{3.39}$$

The first equality follows from an application of Lemma 2.5 to the vector $\mathbf{v}'_i = (b_1^{n_i})^{-1}u_b \in \mathfrak{r}_{S^{n_i}b}$ together with the observation that $aS^{n_i}b \in B_0$ which uses our assumption that a belongs to the set measured in ($\star$). The approximation part in (3.39) comes from the fact that $a \in \operatorname{supp} \mu_{\mathbf{s},U}^{\otimes n_i} = A_{\mathbf{s},U}^{n_i}$ satisfies $\mathrm{E}_n(aS^{n_i}b)\,\mathrm{E}_{n_i}^{-1}(b) \in U$.

We thus conclude from (3.36), (3.38) and (3.39) that for $\beta^{X,U}$-almost every $\mathbf{s} \in K'$,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : t_{n_i}(b, v_i) \asymp \|a_1^{n_i}(b_1^{n_i})^{-1}v_i\|\}) \geq 1 - 3\delta \quad \text{for all } i \gg 1.$$

Taking into account (3.33) we see that for $\beta^{X,U}$-almost every $\mathbf{s} \in K'$,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : \|a_1^{n_i}(b_1^{n_i})^{-1}v_i)\| \asymp \epsilon\}) > 1 - 3\delta \quad \text{for all } i \gg 1. \tag{$\star\star$}$$

Equation ($\star\star$) will take care of **ED2**.

We now turn to **ED3**. Fix $i \gg 1$ and let $k \in \mathbb{N}$. We apply Lemma 3.11 to the representation $V = \wedge^4 \mathfrak{g}$ with $\rho = 1/k$, the weight $\omega_{\mathfrak{l}_0} \in \mathcal{H}_{\mathfrak{z}}(V)$, the vector $\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i)$ where $\mathbf{v}_i$ is as in (3.35) and the flag $\eta = \xi(S^{n_i}b) \in H/P$. The statement of Lemma 3.11 in this case and in particular equation (3.15), implies that: For $\beta^{X,U}$-almost any $\mathbf{s} \in K'$,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : \mathrm{d}_{\mathbb{P}V}(a_1^{n_i}\mathbb{R}\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i), a_1^{n_i}V_{S^{n_i}b}[\omega_{\mathfrak{l}_0}]) < 1/k\}) > 1 - \delta \quad \text{for all } i \gg 1. \tag{3.40}$$

If we also know that if $aS^{n_i}b \in B_0$, which happens whenever a is in the set measured in ($\star$), then we have the equality $a_1^{n_i} V_{S^{n_i}b}[\omega_{\mathfrak{l}_0}] = V_{aS^{n_i}b}[\omega_{\mathfrak{l}_0}]$ and thus we conclude from (3.40) that for $\beta^{X,U}$-almost every $\mathbf{s} \in K'$,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : \mathrm{d}_{\mathbb{P}V}(a_1^{n_i}\mathbb{R}\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i), V_{aS^{n_i}b}[\omega_{\mathfrak{l}_0}]) < 1/k\}) > 1 - 3\delta \quad \text{for all } i \gg 1. \tag{3.41}$$

Next we replace in (3.41) the vector $\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i)$ by $\mathbf{v}_i$. To justify this passage we apply Lemma 3.12 to the representation V with $\rho = 1/k$, $\omega_{\mathbf{m}} = \omega_{\mathfrak{l}_0}$ and the vector $\mathbf{v}_i$. Note that Lemma 3.12 is applicable in light of Lemma 2.7. The statement of Lemma 3.12, in particular equation (3.16), implies that: For $\beta^{X,U}$-almost every $\mathbf{s} \in K'$,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : \mathrm{d}_{\mathbb{P}V}(a_1^{n_i}\mathbb{R}\mathbf{v}_i, a_1^{n_i}\mathbb{R}\tau_{\omega_{\mathfrak{l}_0}}(\mathbf{v}_i)) < 1/k\}) > 1 - \delta \quad \text{for all } i \gg 1. \tag{3.42}$$

Equations (3.41) and (3.42) and the triangle inequality imply that: For $\beta^{X,U}$-almost every $\mathbf{s} \in K'$, for any positive integer k,

$$\mu_{\mathbf{s},U}^{\otimes n_i}(\{a \in A^{n_i} : \mathrm{d}_{\mathbb{P}V}(a_1^{n_i}\mathbb{R}\mathbf{v}_i, V_{aS^{n_i}b}[\omega_{\mathfrak{l}_0}]) < 2/k\}) > 1 - 4\delta \quad \text{for all } i \gg 1. \tag{$\star\star\star$}$$

We then choose $\{i_k\}_{k\in\mathbb{N}}$ with $i_k \to \infty$, such that ($\star\star\star$) holds.

To tie things up and finish this part of the proof we note that for $k \gg 1$, equations ($\star$), ($\star\star$) and ($\star\star\star$) hold for $i = i_k$. Since $\delta < 1/10$, we deduce that for $\beta^{X,U}$-almost every $\mathbf{s} \in K'$ there must exist $n_i \in \mathbb{N}$ and $a_i \in A_{\mathbf{s},U}^{n_i}$ such that properties **ED1**-**ED3** are satisfied and so $\mathbf{s}$ satisfies hypothesis ED. This concludes the proof of Step 1.

Proof of Step 2. Let $\mathbf{s} \in K'$ satisfy hypothesis ED with respect to the unstable sequence $\{v_i\}_{i\in\mathbb{N}}$ and let $\epsilon > 0$ be arbitrarily small. Let $\{n_i\}_{i\in\mathbb{N}}$ and $a_i \in A_{\mathbf{s},U}^{n_i}$ be such that $n_i \to \infty$ as $i \to \infty$ and properties **ED1**-**ED3** hold.

By taking a subsequence if necessary and using **ED1** we may assume that

$$\lim_{i\to\infty} \mathbf{s}(a_i) =: \mathbf{r}_1 \in K \quad \text{and} \quad \lim_{i\to\infty} \mathbf{s}_i(a_i) =: \mathbf{r}_2 \in K.$$

We claim that the relation (3.31) between $\mathbf{s}(a_i)$ and $\mathbf{s}_i(a_i)$ limits to the fact that

$$\text{there exists } w \in \mathfrak{u}_0 \text{ such that } \|w\| \asymp \epsilon \text{ and } \Phi_w(\mathbf{r}_1) = \mathbf{r}_2. \tag{3.43}$$

We prove (3.43): The B-coordinate of $\mathbf{s}(a_i)$ and $\mathbf{s}_i(a_i)$ equals to $a_iS^{n_i}b$ and converges to the B-coordinate of $\mathbf{r}_1$ and $\mathbf{r}_2$ which we denote by $a \in B_0$ (note that this time a is an infinite sequence). Let us denote for $c \in B_0$ by $\mathfrak{m}_c$ the orthogonal complement of $\mathfrak{r}_c$ in $\mathfrak{g}$ and by $\Pi_c : \mathfrak{g}_c \to \mathfrak{m}_c$ the orthogonal projection. Recall that $\tilde{v}_i \in \mathfrak{g}_b$ denotes a representative of v_i. With this notation the relation (3.31) may be rewritten as

$$\mathbf{s}_i(a_i) = \exp(\Pi_{a_iS^{n_i}b}((a_i)_1^{n_i}(b_1^{n_i})^{-1}\tilde{v}_i))\mathbf{s}(a_i). \tag{3.44}$$

Property **ED2** says that

$$\|\Pi_{a_iS^{n_i}b}((a_i)_1^{n_i}(b_1^{n_i})^{-1}\tilde{v}_i)\| = \|(a_i)_1^{n_i}(b_1^{n_i})^{-1}v_i\| \asymp \epsilon.$$

Note that $\Pi_{a_iS^nb}((a_i)_1^{n_i}(b_1^{n_i})^{-1}\tilde{v}_i) \in \mathfrak{m}_{a_iS^{n_i}b}$ and that projectively $\mathfrak{m}_{a_iS^{n_i}b} \to \mathfrak{m}_a$ because of the continuity of $\mathbf{s}' \mapsto \mathfrak{m}_{b'}$ on K, which follows from the continuity of $\mathbf{s}' \mapsto \xi_{b'}$ on K guaranteed by Lemma 3.13. Thus, after taking a subsequence if necessary we get

$$\lim_{i\to\infty} \Pi_{a_iS^{n_i}b}((a_i)_1^{n_i}(b_1^{n_i})^{-1}\tilde{v}_i) = \tilde{v} \in \mathfrak{m}_a \quad \text{where } \|\tilde{v}\| \asymp \epsilon.$$

Equation (3.44) thus limits to the fact that

$$\mathbf{r}_2 = \exp(\tilde{v})\mathbf{r}_1. \tag{3.45}$$

In fact, due to **ED3**, the aforementioned continuity and (3.32) we have that

$$\begin{aligned}\mathbb{R}\tilde{v} \wedge \mathfrak{r}_a &= \lim_{i\to\infty} \mathbb{R}\Pi_{a_i S^{n_i} b}((a_i)_1^{n_i}(b_1^{n_i})^{-1}\tilde{v}_i) \wedge \mathfrak{r}_{a_i S^{n_i} b} \\ &= \lim_{i\to\infty} \mathbb{R}(a_i)_1^{n_i}(b_1^{n_i})^{-1}\tilde{v}_i \wedge \mathfrak{r}_{a_i S^n b} \\ &= \lim_{i\to\infty} (a_i)_1^{n_i}(b_1^{n_i})^{-1}(\tilde{v}_i \wedge \mathfrak{r}_b) \in \lim_{i\to\infty} (\wedge^4\mathfrak{g})_{a_i S^n b}[\omega_{\mathfrak{l}_0}] = (\wedge^4\mathfrak{g})_a[\omega_{\mathfrak{l}_0}].\end{aligned}$$

By part (4) of Lemma 2.7 we deduce that $v \in \mathfrak{l}_a$. Since the map $\mathfrak{m}_a \cap \mathfrak{l}_a \to \mathfrak{u}_a = \mathfrak{l}_a/\mathfrak{r}_a$ is an isometry and since the image of s in H/N is compact the map $\mathrm{Ad}_{s(\xi(a))} : \mathfrak{u}_0 \to \mathfrak{u}_a$ is an isomorphism of bounded norm. It follows that if we denote by $w \in \mathfrak{u}_0$ the image of $\tilde{v}$ then $\|w\| \asymp \epsilon$ and by the definition of the horocyclic flow given in (3.2), equation (3.45) transforms into (3.43).

After establishing the alignment (3.43) we arrive at the endgame. By (3.27) for all $i \in \mathbb{N}$,

$$(\beta^{X,U})^{\Phi}_{\mathbf{s}(a_i)} = (\beta^{X,U})^{\Phi}_{\mathbf{s}} \quad \text{and} \quad (\beta^{X,U})^{\Phi}_{\mathbf{s}_i(a_i)} = (\beta^{X,U})^{\Phi}_{\mathbf{s}_i}$$

because $\widehat{T}^{n_i}(\mathbf{s}) = \widehat{T}^{n_i}(\mathbf{s}(a_i))$ and $\widehat{T}^{n_i}(\mathbf{s}_i) = \widehat{T}^{n_i}(\mathbf{s}_i(a_i))$. Since the LWM-map is continuous on K as it is the output of Lemma 3.13, we can take limits in the above and get that

$$(\beta^{X,U})^{\Phi}_{\mathbf{s}} = (\beta^{X,U})^{\Phi}_{\mathbf{r}_1} = (\beta^{X,U})^{\Phi}_{\mathbf{r}_2}. \tag{3.46}$$

Property **P4** of the LWM-map and equations (3.46) and (3.43) imply $w \in \mathrm{Stab}_{\mathfrak{u}_0}([(\beta^{X,U})^{\Phi}_{\mathbf{s}}])$. Since the latter is a closed subgroup of $\mathfrak{u}_0 \simeq \mathbb{R}$ and ϵ is arbitrarily small we deduce that $\mathrm{Stab}_{\mathfrak{u}_0}([(\beta^{X,U})^{\Phi}_{\mathbf{s}}]) = \mathfrak{u}_0$ which concludes the proof of Step 2 and by that the proof of Claim 3.17. $\square$

4. Proof of Theorem 2.1(c)

Proof of Theorem 2.1(c). Assume that we are in Case II and that $\nu \in \mathcal{P}_\mu(X)$ is μ-ergodic and not the natural lift. Then, according to Theorem 2.1(a)

$$\beta(\{b \in B : \nu_b \text{ has atoms}\}) > 0.$$

The equivariance of the ν_b's and the ergodicity of the shift map imply that the above set has measure 1. Similarly, if $w(b)$ denotes the maximal weight of an atom of ν_b then the equivariance implies that $w = w(b)$ is constant β-almost surely. The same equivariance implies that $\{(b,x) \in B^X : \nu_b(\{x\}) = w\}$ is T-invariant and since it is of positive β^X-measure, it must be of measure 1 by ergodicity of T. That is to say, for β-almost every $b \in B$ the limit measure ν_b is purely atomic and gives the same mass w to each of its atoms. Since ν_b is a probability measure we deduce that there exists $k \in \mathbb{N}$ such that $w = 1/k$ and ν_b has exactly k atoms. By Proposition 2.2 we also know that β-almost surely $\nu_b \in \mathcal{P}(\pi^{-1}(p_b))$.

Under the assumption that Γ is discrete and Zariski dense in $\mathrm{SO}(Q)(\mathbb{R})$ we have by [Fur02, Theorem 2.21] (see also [Led85, Kai00, Kai85]) that the Furstenberg measure $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ on $\mathrm{Gr}_2(\mathbb{R}^3)$ is the Poisson boundary of (Γ, μ). Moreover, if μ is absolutely continuous with respect to the Haar measure on $\mathrm{SO}(Q)(\mathbb{R})$ and contains the identity in the interior of its support then the same conclusion follows from [Fur02, Theorem 2.17] (see also [Fur63b, Theorem 5.3]). By combining [Fur02, Proposition 2.25, Theorem 2.31 parts (a) and (b)] this implies that any extension of the Furstenberg measure is a measure preserving extension. We disintegrate ν into a collection of measures $\{\nu_p\}_{p \in \mathrm{Gr}_2(\mathbb{R}^3)}$ with respect to the map π as in Definition 1.5. Since we have established that ν is a measure preserving extension of $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$

the composition map $b \mapsto p_b \mapsto \nu_{p_b}$ is equivariant. Using the fact that $(b \mapsto p_b)_* \beta = \bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ we have

$$\nu = \int_{\mathrm{Gr}_2(\mathbb{R}^3)} \nu_p \mathrm{d}\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}\,(p) = \int_B \nu_{p_b} \mathrm{d}\beta\,(b)\,, \tag{4.1}$$

Since the collection $\{\nu_{p_b}\}_{b\in B}$ is equivariant and the measures $\{\nu_b\}_{b\in B}$ are the unique equivariant collection satisfying $\nu = \int_B \nu_b \mathrm{d}\beta\,(b)$ we deduce that $\nu_b = \nu_{p_b}$ for β-almost every $b \in B$. Thus we have shown that ν is a measure preserving k-extension of $\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$. □

As mentioned in Remark 1.9 the statement of Theorem 1.8 is amplified in the case μ satisfies assumption (b) to the fact that the natural lift is the unique μ-stationary measure. To see this, note that when a $\Gamma_\mu = \mathrm{SO}(Q)(\mathbb{R})$-orbit intersects a fibre of π above a plane in the circle of isotropic planes $\mathcal{C} = \operatorname{supp} \bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ it intersects it in infinitely many points. Thus the possibility of the existence of an ergodic finite extension is excluded.

5. Non-escape of mass

In this section we construct a proper function on X which can be thought of like a height function. We will show that this function is contracted by the averaging operator induced by μ, where μ is as in Case I or Case II. The existence of such a function is important in two ways. First, it implies that almost surely the random walks of μ on X are recurrent in a strong sense. This recurrence will imply that the limiting distribution of almost every random walk is a probability measure or in other words that mass does not escape. In turn this will allow us to conclude Theorem 2.1(d) at the end of this section. Second, this function will also play an important role in the proof of Theorem 2.1(b), given in §6.

5.1. **Replacing μ by μ^{*n_0}.** Before starting the construction of the contracted function we note that the statements in Theorem 2.1 are not affected by replacing μ by μ^{*n_0}. Using Lemma 2.10 we choose $n_0 > 0$ and make the replacement

$$\mu := \mu^{*n_0}$$

so that for some $L_0 > 0$ the following holds:

(1) In both Case I and Case II, for all $v \in \mathbb{R}^3 \smallsetminus \{0\}$ and $w \in \wedge^2\mathbb{R}^3 \smallsetminus \{0\}$ one has

$$\int_G \log\left(\frac{\|gv\|}{\|v\|} \Big/ \frac{\|gw\|^{1/2}}{\|w\|^{1/2}}\right) \mathrm{d}\mu\,(g) > L_0. \tag{5.1}$$

(2) In case Case I, for all $p \in \mathrm{Gr}_2(\mathbb{R}^3)$, $u \in \wedge^3\mathfrak{r}_p \smallsetminus \{0\}$ and $v \in \mathfrak{g} \smallsetminus \mathfrak{r}_p$ one has

$$\int_G \log\left(\frac{\|g(v\wedge u)\|}{\|v\wedge u\|} \Big/ \frac{\|gu\|}{\|u\|}\right) \mathrm{d}\mu\,(g) > L_0. \tag{5.2}$$

5.2. **The contraction hypothesis.** Suppose that G acts continuously on a locally compact metric space Y and $\eta \in \mathcal{P}(G)$, then for a measurable $f : Y \to [0,\infty)$ define

$$\mathrm{A}_\eta\, f\,(x) := \int_G f\,(gx)\,\mathrm{d}\eta\,(g)\,.$$

Recall that a function $f : Y \to [0,\infty)$ is said to be *proper* if $f^{-1}\,(C)$ is pre-compact for all compact subsets $C \subset [0,\infty)$. It is said to be *lower semi-continuous* if the sublevel sets $f^{-1}([0,M])$ are closed for any $M \geq 0$.

Definition 5.1. *A function $f : Y \to [0, \infty)$ satisfies the* contraction hypothesis *with respect to η on Y if there exist constants $c < 1$ and $b > 1$ such that*

$$\mathrm{A}_\eta f(y) \le cf(y) + b \quad \text{for all } y \in Y$$

We use the notation $\mathrm{CH}_\eta(Y)$ for the set of all such functions.

Our next goal is to construct a function $f \in \mathrm{CH}_\mu(X)$. The idea constructing contracted functions in order to establish some kind of recurrence can be traced back to the paper [EM04] of A. Eskin and G. Margulis. These ideas were later taken up and used by Benoist and Quint in [BQ12], [BQ11] and [BQ13b]. The following lemma is an extension of [BQ13b, Lemma 6.12] which in turn is an extension of [EM04, Lemma 4.2]. But first we introduce a definition.

Definition 5.2. *Let $\mathcal{F}$ be a family of positive functions on G and $\eta \in \mathcal{P}(G)$ such that:*
(1) There exist $\delta_0 > 0$ and $0 \le I_0 < \infty$ such that

$$\int_G \sup_{f\in\mathcal{F}} f(g)^{\delta_0} \,\mathrm{d}\eta(g) \le I_0.$$

(2) There exists $L_0 > 0$ such that

$$\inf_{f\in\mathcal{F}} \int_G \log f(g) \,\mathrm{d}\eta(g) \ge L_0.$$

Then, we say that $\mathcal{F}$ is uniformly (δ_0, I_0, L_0)-expanded *by η.*

Using this definition we prove a very mild generalisation of [BQ13b, Lemma 6.12]. The proof is identical to the one given there.

Lemma 5.3. *Let $\eta \in \mathcal{P}(G)$ and (δ_0, I_0, L_0) be positive parameters. Let $\mathcal{F}$ be a family of positive functions on G uniformly (δ_0, I_0, L_0)-expanded by η. Then there exists $\delta_1 = \delta(\delta_0, I_0, L_0) > 0$ such that for all $0 < \delta \le \delta_1$ there exists $0 < c = c(\delta, L_0) < 1$ such that for all $f \in \mathcal{F}$ one has*

$$\int_G f(g)^{-\delta} \,\mathrm{d}\eta(g) \le c.$$

Proof. Set $\delta_1 := \min\left\{\frac{\delta_0}{2}, \frac{L_0\delta_0^2}{4I_0}\right\}$ and let $\eta \in \mathcal{P}(G)$ be such that $\mathcal{F}$ is uniformly (δ_0, I_0, L_0)-expanded according to Definition 5.2. We will use the facts that

$$\exp(x) \le 1 + x + \frac{x^2}{2}\exp(|x|) \quad \text{and} \quad x^2 \le \exp(|x|)$$

for all $x \in \mathbb{R}$. Then for any $f \in \mathcal{F}$ and $\delta \in \mathbb{R}$ we have

$$\begin{aligned}\int_G f(g)^{-\delta} \,\mathrm{d}\eta(g) &= \int_G \exp(-\delta \log f(g)) \,\mathrm{d}\eta(g) \\ &\le 1 - \delta \int_G \log f(g) \,\mathrm{d}\eta(g) + \frac{\delta^2}{2} \int_G (\log f(g))^2 f(g)^\delta \,\mathrm{d}\eta(g)\end{aligned}$$

and

$$(\log f(g))^2 \le \frac{4}{\delta_0^2} f(g)^{\delta_0/2}.$$

Using these inequalities together with conditions (1) and (2) of Definition 5.2 and our choice of δ_1, we see that for all $f \in \mathcal{F}$ and $0 < \delta < \delta_1$ one has

$$\int_G f(g)^{-\delta} \,\mathrm{d}\eta(g) \le 1 - \delta L_0 + 2\frac{\delta^2}{\delta_0^2} I_0 \le 1 - \frac{\delta}{2} L_0 < 1,$$

so the statement holds with $c = 1 - \frac{\delta}{2} L_0$ as required. □

Remark 5.4. *It will be important for us that given $\eta \in \mathcal{P}(G)$ and a family of positive functions $\mathcal{F}$ uniformly (δ_0, I_0, L_0)-expand by η the constants δ_1 and c whose existence is assured by Lemma 5.3 are uniform over all measures in the set*

$$\{\eta \in \mathcal{P}(G) : \mathcal{F} \textit{ is uniformly } (\delta_0, I_0, L_0)\textit{-expanded by } \eta\}.$$

Let Λ denote a 2-lattice in $\mathbb{R}^3$ and let $[\Lambda] \in X$ denote the corresponding homothety class. We denote by $|\Lambda|$ the co-volume of Λ in the plane it spans. For any $v \in \Lambda$ we define the *normalised length* of v with respect to Λ to be

$$N_\Lambda(v) := \frac{\|v\|}{|\Lambda|^{1/2}}.$$

This quantity already appeared implicitly in the proof of Proposition 2.4. We let,

$$f_{\Lambda,v}(g) := \frac{N_{g\Lambda}(gv)}{N_\Lambda(v)}$$

That is, $f_{\Lambda,v}(g)$ is the cocycle that measures by which factor v is stretched under the action of g taking into account the normalisation factors which make Λ and $g\Lambda$ of co-volume 1 in their respective planes.

Let

$$\mathcal{F} := \{f_{\Lambda,v}\}_{[\Lambda]\in X, v\in\Lambda}.$$

The main step towards constructing a function in $\mathrm{CH}_\mu(X)$ is the following.

Proposition 5.5. *Let μ be as in Case I or Case II and suppose that (5.1) hold. Then $\mathcal{F}$ is uniformly (δ_0, I_0, L_0)-expanded by μ for some L_0 as in (5.1) and some positive δ_0 and I_0.*

Proof. We verify conditions (1), (2) of Definition 5.2. The validity of condition (1) is immediate with say $\delta_0 = 1$ and some $I_0 < \infty$ from the assumption that μ is compactly supported. For condition (2) we note that if $\Lambda = \mathrm{span}_{\mathbb{Z}}\{u, w\}$, then

$$f_{\Lambda,v}(g) = \frac{\|gv\|}{\|g(u\wedge w)\|^{1/2}} \Big/ \frac{\|v\|}{\|u\wedge w\|^{1/2}}$$

for all $v \in \Lambda$. It follows that condition (2) is implied by equation (5.1) which holds for μ as indicated in §5.1. □

For $[\Lambda] \in X$ we set

$$u_X([\Lambda]) := \Big(\min_{v\in\Lambda\smallsetminus\{0\}} N_\Lambda(v)\Big)^{-1}.$$

It is clear from the definition of N_Λ that u_X is well defined in the sense that its value does not depend on the choice of Λ from $[\Lambda]$. Moreover, by Mahler's compactness criterion that $u_X : X \to [0,\infty)$ is a continuous proper function. The following proposition establishes the existence of a function which satisfies the contraction hypothesis of Definition 5.1.

Proposition 5.6. *Let μ be as in Case I or Case II and suppose that (5.1) and (5.2) hold. Then, for all δ sufficiently small $u_X^\delta \in \mathrm{CH}_\mu(X)$.*

Proof. Given $M > 0$ we split X into $X^{\le M} = u_X^{-1}([0, M])$ and $X^{>M} = u_X^{-1}((M,\infty))$. We claim that there exists $M > 0$ such that if $[\Lambda] \in X^{>M}$ then there exists a unique (up to sign) vector $v_{\min}(\Lambda) \in \Lambda$ such that $u_X([\Lambda]) = N_\Lambda(v_{\min}(\Lambda))$ and $u_X(g[\Lambda]) = N_{g\Lambda}(gv_{\min}(\Lambda))$ for all $g \in \mathrm{supp}\,\mu$. First we note that because we are dealing with 2-lattices for any $M \ge 1$ and $[\Lambda] \in X^{>M}$ the vector $v_{\min}(\Lambda)$ is well defined up to sign. Now set $M := \sup_{g\in\mathrm{supp}\,\mu}\|g\|^2$

and suppose there exists $v' \in g\Lambda$ with $v' \neq gv_{\min}(\Lambda)$ and such that $u_X([\Lambda]) = N_{g\Lambda}(v')$. Then note that

$$\frac{\|g^{-1}v'\|}{\|g\| |g\Lambda|^{1/2}} \leq N_{g\Lambda}(v') \leq N_{g\Lambda}(gv_{\min}(\Lambda)) \leq \frac{\|g\| \|v_{\min}(\Lambda)\|}{|g\Lambda|^{1/2}}.$$

Since $g^{-1}v'$ and $v_{\min}$ cannot be colinear we see that this is a contradiction if $\|v_{\min}(\Lambda)\| < 1/M$ since in this case we would get that Λ contains two non colinear vectors with norm less than 1.

Next suppose that $[\Lambda] \in X^{>M}$ and write $\Lambda = \operatorname{span}_{\mathbb{Z}}\{u, w\}$ so that for $\delta > 0$ one has

$$\mathrm{A}_\mu\, u_X^\delta([\Lambda]) = \int_G u_X^\delta(g[\Lambda]) \mathrm{d}\mu\,(g) = \int_G \left(\frac{\|g(u \wedge w)\|^{1/2}}{\|gv_{\min}(\Lambda)\|} \right)^\delta \mathrm{d}\mu\,(g)\,.$$

By Proposition 5.5 and Lemma 5.3, if δ is small enough, there exists $0 < c < 1$ such that

$$\int_G \left(\frac{\|g(u \wedge w)\|^{1/2}}{\|gv_{\min}(\Lambda)\|} \right)^\delta \mathrm{d}\mu\,(g) \leq c \left(\frac{\|(u \wedge w)\|^{1/2}}{\|v_{\min}(\Lambda)\|} \right)^\delta = u_X^\delta([\Lambda]).$$

Fix δ and let $0 < c < 1$ be such a number. If $[\Lambda] \in X^{\leq M}$ then from the compactness of $\operatorname{supp}\mu$ and the properness u_X^δ we conclude that $\mathrm{A}_\mu\, u_X^\delta([\Lambda]) \leq b(M, \delta, \mu) = b$. In any case we have

$$\mathrm{A}_\mu\, u_X^\delta([\Lambda]) \leq c u_X^\delta([\Lambda]) + b,$$

that is, $u_X^\delta \in \mathrm{CH}_\mu(X)$ as desired. $\square$

Using the existence of a proper function in $\mathrm{CH}_\mu(X)$ we can give a proof of Theorem 2.1(d) by citing Benoist and Quint.

Proof of Theorem 2.1(d). Let $x \in X$ be given. By [BQ12, Corollary 2.2], any weak-* accumulation point of the sequence $\frac{1}{n}\sum_{k=1}^n \mu^{*k} * \delta_x$ is a probability measure on X. It is also evidently μ-stationary.

Moreover, it follows from [BQ13a, Corollary 3.3] that for all $x \in X$, for β-almost every $b \in B$, any weak-* accumulation point of the sequence $\frac{1}{n}\sum_{k=1}^n \delta_{b_k^1 x}$ is μ-stationary. Thus, we are only left to establish that β-almost surely, such an accumulation point is a probability measure. This is again a consequence of the existence of a function in $\mathrm{CH}_\mu(X)$. Indeed, [BQ13a, Example 3.1, Proposition 3.9] implies this exact statement since the contracted function u_X is proper. $\square$

6. The limit measures are non-atomic

In this section we assume μ is as in Case I and also assume the validity of (5.1), (5.2) as in §5.1 which is ensured by replacing μ by μ^{n_0} if necessary. The main goal of this section is to prove Theorem 2.1(b).

6.1. **Metric considerations.**

We will need to have some understanding of a convenient metric on X. In order to do this we study the local structure of X. For $p \in \mathrm{Gr}_2(\mathbb{R}^3)$ let

$$\Pi_p : \mathfrak{g} \to (\mathfrak{r}_p)^\perp := \mathfrak{m}_p,$$

be the orthogonal projection where the inner product in the above definition is supposed to be K-invariant. It is important to note that Π_p is not equivariant. We use the convention that for any representation V of H the notation $\|g\|_V$ stands for the operator norm of g on V.

Let d_X denote a metric on X induced by a Riemannian metric obtained in the following manner: For a point $x \in X$ the derivative at the identity $d_e\alpha_x$ of the orbit map $\alpha_x : G \to X$,

$g \mapsto gx$ satisfies $\ker d_e\alpha_x = \mathfrak{r}_p$ where p is the plane of x. Since $d_e\alpha_x$ is of full rank, it restricts to a linear isomorphism $d_e\alpha_x : \mathfrak{m}_p = \mathfrak{r}_p^{\perp} \to T_xX$. We use this isomorphism to transport the inner product structure that $\mathfrak{m}_p$ inherits from $\mathfrak{g}$ to T_xX thus inducing a Riemannian metric on X.

If we denote by c_g conjugation by g then for any $x \in X$ we have the following commutative diagram and its derivative:

$$\begin{tikzcd} G \arrow[r, "\alpha_x"] \arrow[d, "c_g"'] & X \arrow[d, "g"] \\ G \arrow[r, "\alpha_{gx}"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{g} \arrow[r, "d_e\alpha_x"] \arrow[d, "\mathrm{Ad}_g"'] & T_xX \arrow[d, "d_xg"] \\ \mathfrak{g} \arrow[r, "d_e\alpha_{gx}"'] & T_xX \end{tikzcd}$$

The fact that the horizontal maps on the right diagram are linear surjections of norm at most 1 implies that the norm of $d_xg : T_xX \to T_{gx}X$ is bounded by $\|g\|_{\mathfrak{g}}$. In particular, since the metric d_X is defined in terms of length of paths, it satisfies the important inequality

$$\mathrm{d}_X(gx, gy) \le \|g\|_{\mathfrak{g}}\, \mathrm{d}_X(x, y) \quad \text{for all } g \in G \text{ and } x, y \in X. \tag{6.1}$$

Consider $X \times \mathfrak{g}$ as a Riemannian manifold and consider $X^* := \{(x, v) : v \in \mathfrak{m}_{\pi(x)}\}$ as a submanifold. The map $\psi : X^* \to X \times X$ defined by $\psi(x, v) = (x, \exp(v)x)$ is smooth and has the property that on the submanifold $X_0^* = \{(x, 0) \in X^*\}$ the derivative

$$d_{(x,0)}\psi : T_xX \oplus \mathfrak{m}_{\pi(x)} \to T_xX \oplus T_xX$$

is an isometry. In fact, it equals the identity after identifying T_xX with $\mathfrak{m}_{\pi(x)}$ as described earlier. Since X^* and $X \times X$ are of the same dimension we conclude that there is an open neighbourhood $X_0^* \subset \mathcal{V}_0 \subset X^*$ that is mapped by ψ diffeomorphically onto an open neighbourhood $\Delta_X = \psi(X_0^*) \subset \mathcal{U}_0 \subset X \times X$. Given $(x, y) \in \mathcal{U}_0$ we define the *orthogonal displacement vector* $o_{x,y}$ between x and y to be the unique vector $v \in \mathfrak{m}_{\pi(x)}$ such that $(x, v) \in \mathcal{V}_0$ and $\psi(x, v) = (x, y)$, or in other words $y = \exp(v)x$. We prove the following.

Lemma 6.1. *For any compact set $E \subset G$ and all $0 < c < 1$ there exists a neighbourhood of the diagonal $\mathcal{U} \subset X \times X$ such that for all $(x, y) \in \mathcal{U}$ and $g \in E \cup E^{-1} \cup \{e\}$ one has:*

(1) The orthogonal displacement $o_{gx,gy}$ is well defined.
(2) The following inequality holds $c\|o_{gx,gy}\| \le \mathrm{d}_X(gx, gy) \le c^{-1}\|o_{gx,gy}\|$.
(3) For all $u \in \wedge^3\mathfrak{r}_{\pi(x)} \smallsetminus \{0\}$ one has $c\|o_{gx,gy}\| \le \|g(o_{x,y} \wedge u)\|/\|gu\| \le c^{-1}\|o_{gx,gy}\|$.

Proof. Throughout the proof we may assume that $E = E \cup E^{-1} \cup \{e\}$ by enlarging it if necessary. First we prove (1). Let $\mathcal{U}_0$ be the neighbourhood of Δ_X on which the orthogonal displacement is defined. We first show that $\cap_{g\in E}\, g\mathcal{U}_0$ contains a neighbourhood of Δ_X. This is done by showing that for any compact $K \subset X$ we have

$$\mathrm{d}_{X\times X}(K \times K \smallsetminus \cap_{g\in E}\, g\mathcal{U}_0, \Delta_X) > 0. \tag{6.2}$$

To this end, let $K \subset X$ be a compact set. By (6.1) and the compactness of E, we deduce from the fact that

$$\mathrm{d}_{X\times X}((K \times K \smallsetminus \mathcal{U}_0), \Delta_X) > 0,$$

that

$$\mathrm{d}_{X\times X}(\cup_{g\in E}\, g(K \times K \smallsetminus \mathcal{U}_0), \Delta_X) > 0.$$

However, since $K \times K \smallsetminus \cap_{g\in E}\, g\mathcal{U}_0 \subset \cup_{g\in E}\, g(K \times K \smallsetminus \mathcal{U}_0)$ the previous equation implies (6.2) as claimed.

We conclude that $\cap_{g\in E} g\mathcal{U}_0$ contains a neighbourhood $\mathcal{U}_1$ of Δ_X and deduce that for any $g \in E$ and $(x,y) \in \mathcal{U}_1$, $(gx,gy) \in \mathcal{U}_0$ so that the orthogonal displacement $o_{gx,gy}$ is well defined.

Now we will prove (2). Let $0 < c < 1$ be given and for $A, B \in \mathbb{R}$ write $A \sim_c B$ to denote $cA < B < c^{-1}A$. Consider the map $\psi : X^* \to X \times X$ and the neighbourhood $\mathcal{V}_0$ as defined before the statement of the lemma. Let $\mathcal{V}_1 := \psi^{-1}(\mathcal{U}_1) \subset \mathcal{V}_0$. Since the differential $d\psi$ is an isometry on the submanifold X_0^* and E is compact there is a neighbourhood $\mathcal{V}_2 \subset \mathcal{V}_1$ of X_0^* such that for all $(x,v) \in \mathcal{V}_2$ and $g \in E$ one has $\|d_{(gx,gv)}\psi^{\pm 1}\| \sim_c 1$. The image $\mathcal{U}_2 := \psi(\mathcal{V}_2) \subset \mathcal{U}_1$ is then a neighbourhood of Δ_X. Next we replace $\mathcal{V}_2$ and $\mathcal{U}_2$ by even smaller neighbourhoods $\mathcal{V}_3$ and $\mathcal{U}_3 := \psi(\mathcal{V}_3)$ of X_0^* and Δ_X respectively, so that for all $(x,v) \in \mathcal{V}_3$ the whole interval $\{(x,tv) : t \in [0,1]\}$ is contained in $\mathcal{V}_2$. Similarly, for any $(x,y) \in \mathcal{U}_3$, the geodesic path between (x,x) and (x,y) is contained in $\mathcal{U}_2$.

Given $(x,y) \in \mathcal{U}_3$ let $o_{x,y}$ denote the corresponding orthogonal displacement so that $\psi(x, o_{x,y}) = (x,y)$. Since the path $\zeta(t) = (x, to_{x,y})$ is the geodesic in X^* from $(x,0)$ to $(x, o_{x,y})$ and is of length $\|o_{x,y}\|$ and since it is contained in $\mathcal{V}_2$ on which $\|d\psi\| \sim_c 1$, we conclude that the image path $\psi(\zeta(t))$ connecting (x,x) to (x,y) has length $\sim_c \|o_{x,y}\|$. But, the distance in $X\times X$ from (x,x) to (x,y) is exactly $\mathrm{d}_X(x,y)$ and so we obtain the inequality $\mathrm{d}_X(x,y) < c^{-1}\|o_{x,y}\|$ for all $(x,y) \in \mathcal{U}_3$ and $g \in E$.

For the other inequality, let $(x,y) \in \mathcal{U}_3$, $g \in E$ and let $\zeta(t)$ denote the geodesic path between (x,x) to (x,y) which is of length $\mathrm{d}_X(x,y)$ as mentioned earlier. By the choice of $\mathcal{U}_3$, $\zeta(t) \in \mathcal{U}_2$ for all t and therefore, on applying $\psi|_{\mathcal{U}_2}^{-1}$ we obtain a path connecting $(x,0)$ and $(x, o_{x,y})$ whose length is $< c^{-1}\,\mathrm{d}_X(x,y)$. Since the distance between $(x,0)$ and $(x,o_{x,y})$ is $\|o_{x,y}\|$ we obtain the inequality $\|o_{x,y}\| < c^{-1}\,\mathrm{d}_X(x,y)$. In total we showed that for all $(x,y) \in \mathcal{U}_3$ and $g \in E$ one has $\mathrm{d}_X(x,y) \sim_c \|o_{x,y}\|$. To finish, we replace $\mathcal{U}_3$ by $\mathcal{U}_4$ a neighbourhood of Δ_X contained in $\cap_{g\in E} g\mathcal{U}_3$ (in a similar fashion to the proof of part (1)) and conclude that for all $(x,y) \in \mathcal{U}_4$ and all $g \in E$ we have that $(gx,gy) \in \mathcal{U}_3$ and therefore $\mathrm{d}_X(gx,gy) \sim_c \|o_{gx,gy}\|$ as desired

Finally we prove (3). Let $\mathcal{U} = \mathcal{U}_4$ be as in the proof of part (2) and let $(x,y) \in \mathcal{U}$. Note that $\|g(o_{x,y} \wedge u)\|/\|gu\| = \|\Pi_{gx}(go_{x,y})\|$ for all $u \in \wedge^3 \mathfrak{r}_{\pi(x)} \smallsetminus \{0\}$ and that both of $\exp(o_{gx,gy})$ and $\exp(go_{x,y})$ take gx to gy so

$$\exp(-go_{x,y})\exp(o_{gx,gy}) \in \mathrm{Stab}_G(gx).$$

There is a neighbourhood of the identity $\mathcal{L}_0$ in G such that $\log : \mathcal{L}_0 \to \mathfrak{g}$ is well defined. By shrinking $\mathcal{U}$ if necessary we may suppose that the above product is in $\mathcal{L}_0$ for $g \in E$ and $(x,y) \in \mathcal{U}$. Therefore, we may apply the logarithm and see that the result lies in $\ker \Pi_{gx} = \mathfrak{r}_{\pi(gx)}$ (which is the Lie algebra of $\mathrm{Stab}_G(gx)$). On the other hand [Tao14, §2] we have

$$\log(\exp(-go_{x,y})\exp(o_{gx,gy})) = o_{gx,gy} - go_{x,y} + O(\|o_{gx,gy}\|\|o_{x,y}\|\|g\|_{\mathfrak{g}}).$$

Applying the projection Π_{gx} we see that

$$o_{gx,gy} = \Pi_{gx}(go_{x,y}) + O(\|o_{gx,gy}\|\|o_{x,y}\|\|g\|_{\mathfrak{g}}). \tag{6.3}$$

Equation (6.3) together with part (2) imply that on shrinking $\mathcal{U}$ if necessary, the ratio $\|\Pi_{gx}(go_{x,y})\|/\|o_{gx,gy}\|$ is bounded away from zero for $(x,y) \in \mathcal{U}$ and $g \in E$. In equation (6.3) we take norms, use the triangle inequality and divide by $\|\Pi_{gx}(go_{x,y})\|$ to arrive at

$$\|o_{gx,gy}\| \Big/ \frac{\|g(o_{x,y} \wedge u)\|}{\|gu\|} = \frac{\|o_{gx,gy}\|}{\|\Pi_{gx}(go_{x,y})\|} = 1 + O(\|o_{x,y}\|\|g\|_{\mathfrak{g}}),$$

where the cancellation in the big-O is justified by the aforementioned boundedness away from zero of $\|\Pi_{gx}(go_{x,y})\|/\|o_{gx,gy}\|$. Now it is clear that since $g \in E$ and E is compact, if $\mathcal{U}$

is chosen small enough then the big-O in the above equality is as small as we wish yielding part (3) of the proposition. $\square$

6.2. **A criterion for non-atomicity of the limit measures.** In this section we leave for a moment the space X and work in an abstract setting. We follow closely [BQ13b, §6]. We assume throughout that we are working in the following setting:

S1: Y is a locally compact metric space on which G acts.
S2: There exists a proper lower semi-continuous contracted function $u_Y \in \mathrm{CH}_\mu(Y)$.

Remark 6.2. *In §6.3 we will apply the results of this section to $Y = X \times X$ with the function $u_{X\times X}(x,y) := u_X^\delta(x) + u_X^\delta(y) \in \mathrm{CH}_\mu(X\times X)$, where $\delta > 0$ is small enough so that $u_X^\delta \in \mathrm{CH}_\mu(X)$ by Proposition 5.6.*

For $M > 0$ we consider the compact set

$$Y_M := \{y \in Y : u_Y(y) \leq M\}.$$

For $y \in Y_M$ and $b \in B$ we define the stopping time

$$\rho_{M,y}(b) := \inf\left\{n \geq 1 : b_n^1 y \in Y_M\right\}$$

and refer to it as the *first return time* to Y_M. It is a stopping time in the sense that $\{\rho_{M,y}(b) \leq n\}$ is independent from b_j for any $j > n$. The sets Y_M have remarkable recurrence properties as reflected by the following proposition.

Proposition 6.3. *Suppose that **S1** and **S2** hold. Then for any M large enough there exists $c > 1$ such that*

$$\sup_{y\in Y_M} \int_B c^{\rho_{M,y}(b)} \mathrm{d}\beta(b) < \infty.$$

In particular the first return time is integrable and β-almost surely finite.

Proof. This is [BQ13b, Definition 6.1 and Proposition 6.3]. $\square$

The first return time naturally defines a map $\widehat{\rho}_M : B \times Y \to G$ called the *first return cocycle* where

$$\widehat{\rho}_{M,y}(b) := b^1_{\rho_{M,y}(b)}.$$

In turn, the first return cocycle induces a collection of transition probability measures $\mu_{M,y} \in \mathcal{P}(G)$ which are the images of β by the first return cocycle. In other words for $y \in Y_M$ and $f \geq 0$ a measurable function on G,

$$\int_G f(g)\mathrm{d}\mu_{M,y}(g) := \int_B f(\widehat{\rho}_{M,y}(b))\mathrm{d}\beta(b) = \int_B f(b^1_{\rho_{M,y}(b)})\mathrm{d}\beta(b).$$

Finally the transition probability measures $\mu_{M,y}$ induce a Markov operator called *the first return Markov operator* which is denoted by $\mathrm{A}_{M,\mu}$ and is defined as follows. For any $f \geq 0$ measurable function on Y_M

$$\mathrm{A}_{M,\mu} f(y) := \int_G f(gy)\,\mathrm{d}\mu_{M,y}(g).$$

Extending Definition 5.1 we say that $f \in \mathrm{CH}_{\mathrm{A}_{M,\mu}}(Y_M)$ if $f : Y_M \to [0,\infty]$ is such that there exists constants $c < 1$, $b > 0$ satisfying

$$\mathrm{A}_{M,\mu} f(y) \leq cf(y) + b \quad \text{for all } y \in Y_M.$$

We will prove Theorem 2.1(b) by an application of the following criterion. It shows that one can deduce the non-atomicity of the limit measures of $\nu \in \mathcal{P}_\mu(Y)$, for Y as above, if one can build functions satisfying the contraction hypothesis on bounded parts of $Y \times Y \setminus \Delta_Y$, where Δ_Y denotes the diagonal copy of Y in $Y \times Y$.

Proposition 6.4. *Suppose that **S1** and **S2** hold. Consider the product space $Y \times Y$ on which G acts diagonally and consider the function $u_{Y\times Y}(y_1, y_2) := u_Y(y_1) + u_Y(y_2)$ so that $u_{Y\times Y} \in \mathrm{CH}_\mu(Y \times Y)$. Let $(Y \times Y)_M$ and $\mu_{M,(y_1,y_2)}$ and $\mathrm{A}_{M,\mu}$ be the sublevel sets, transition probability measures and Markov operator associated to the action of G on $Y \times Y$ with respect to $u_{Y\times Y}$.*

If for every large enough M there exists a proper continuous function $v_M : (Y \times Y)_M \setminus \Delta_Y \to [0, \infty)$ such that $v_M \in \mathrm{CH}_{\mathrm{A}_{M,\mu}}((Y \times Y)_M \setminus \Delta_Y)$, then for any atom-free $\nu \in \mathcal{P}_\mu(Y)$, the limit measures are β-almost surely non-atomic

Proof. This follows from [BQ13b, Proposition 6.16 and Proposition 6.17]. □

We continue to collect results from [BQ13b] that will allow us to construct the functions v_M in Proposition 6.4.

Proposition 6.5. *Suppose that **S1** and **S2** hold. Then, if $N : G \to [0, \infty)$ is a continuous submultiplicative function, for any M large enough there exists $\delta > 0$ such that*

$$\sup_{y\in Y_M} \int_G N(g)^\delta \, \mathrm{d}\mu_{M,y}(g) < \infty.$$

Proof. This is [BQ13b, Definition 6.1, Proposition 6.3 and Proposition 6.7]. Notice that we are assuming μ is compactly supported so it has finite exponential moments with respect to N in the terminology of [BQ13b, Definition 6.6]. □

For the following proposition we give a full proof. This is a slight upgrade of [BQ13b, Lemma 6.10] but as far as we could tell the proof there is incorrect.

Proposition 6.6. *Suppose that **S1** and **S2** hold. Let M be large enough so that Proposition 6.3 is applicable and in particular, the stopping times $\{\rho_{M,y}\}_{y\in Y_M}$ are integrable.*

Let G act on a space W and assume that $f : G \times W \to \mathbb{R}$ is an additive cocycle in the sense that $f(gh, w) = f(g, hw) + f(h, w)$ for all $g, h \in G$ and $w \in W$. Assume that:

(1) There exists $J_0 > 0$ such that $\sup_{w\in W} \|f(-, w)\|_{L^\infty(G,\mu)} < J_0$.

(2) There exists $L_0 > 0$ such that $\inf_{w\in W} \int_G f(g, w)\mathrm{d}\mu(g) > L_0$.

Then, for all $w \in W$ and $y \in Y_M$ we have that $f(-, w) \in L^1(G, \mu_{M,y})$ and moreover

$$\inf_{w\in W, y\in Y_M} \int_G f(g, w)\mathrm{d}\mu_{M,y}(g) \geq L_0. \tag{6.4}$$

Proof. Let $\mathcal{B}$ denote the Borel σ-algebra of B and $\rho : B \to \mathbb{N}$ be an integrable stopping time. That is, $\int_B \rho \mathrm{d}\beta < \infty$ and for all $n \in \mathbb{N}$ one has $\{b : \rho(b) \leq n\}$ is measurable with respect to the sub-σ-algebra $\mathcal{B}_n$ of $\mathcal{B}$ generated by the cylinder sets obtained by specifying the first n co-ordinates. Let $\mu_\rho := (b \mapsto b^1_{\rho(b)})_* \beta \in \mathcal{P}(G)$ be the push-forward of β under the almost surely defined product map $b \mapsto b^1_{\rho(b)}$. Then, we will prove that under the assumptions (1) and (2), for all $w \in W$ one has

$$\int_G |f(g, w)|\mathrm{d}\mu_\rho(g) \leq J_0 \int_B \rho \mathrm{d}\beta \quad \text{and} \quad \int_G f(g, w)\mathrm{d}\mu_\rho(g) \geq L_0 \int_B \rho \mathrm{d}\beta. \tag{6.5}$$

If M is as in the statement, the stopping times $\{\rho_{M,y}\}_{y\in Y_M}$ are integrable so the left inequality of equation (6.5) applied with $\rho = \rho_{M,y}$ proves that $f(-, w) \in L^1(G, \mu_{M,y})$ for all $w \in W$ and $y \in Y_M$. Moreover, the right inequality of (6.5) applied with the same choice of ρ implies equation (6.4) because the integral of a stopping time is at least 1.

Fix $w \in W$ and for $i \in \mathbb{N}$ let $X_i(b) := f(b_i, b_{i-1}^1 w)$, where b_0^1 denotes the empty product, so that by the cocycle equation

$$f(b_{\rho(b)}^1, w) = \sum_{i=1}^{\rho(b)} X_i(b).$$

With this notation, we can rewrite equation (6.5) as follows

$$\int_B \Big|\sum_{i=1}^{\rho(b)} X_i(b)\Big| \mathrm{d}\beta\,(b) \leq J_0 \int_B \rho \mathrm{d}\beta \quad \text{and} \quad \int_G \sum_{i=1}^{\rho(b)} X_i(b) \mathrm{d}\beta\,(b) \geq L_0 \int_B \rho \mathrm{d}\beta. \tag{6.6}$$

Note that

$$\Big|\sum_{i=1}^{\rho(b)} X_i(b)\Big| \leq \sum_{i=1}^{\infty} \mathbf{1}_{\{b \in B : \rho(b) \geq i\}}(b) |X_i(b)| \quad \text{for all} \quad b \in B.$$

Hence, using (1) which implies that $|X_i| \leq J_0$ and the monotone convergence theorem, we obtain

$$\begin{aligned}
\int_B \Big|\sum_{i=1}^{\rho(b)} X_i(b)\Big| \mathrm{d}\beta\,(b) &\leq \sum_{i=1}^{\infty} \int_{\{b \in B : \rho(b) \geq i\}} |X_i| \mathrm{d}\beta \\
&\leq \sum_{i=1}^{\infty} \beta(\{b \in B : \rho(b) \geq i\}) J_0 = J_0 \int \rho \mathrm{d}\beta.
\end{aligned}$$

This is the left inequality of (6.6). We now turn to the proof of the right inequality of (6.6). Consider the sequence of random variables $Z_n := \sum_{i=1}^{n} (X_i - L_0)$. Since X_i is $\mathcal{B}_i$-measurable for all $i \in \mathbb{N}$, one has $\mathbb{E}(Z_n | \mathcal{B}_{n-1}) = Z_{n-1} + \mathbb{E}(X_n | \mathcal{B}_{n-1}) - L_0$. Hence, provided that

$$\mathbb{E}(X_n | \mathcal{B}_{n-1}) \geq L_0 \quad \beta\text{-almost surely,} \tag{6.7}$$

Z_n is a submartingale with respect to the filtration $\mathcal{B}_n$. Recall that the definition of conditional expectation is given by integration with respect to the conditional measures. For all $b \in B$ the conditional measure $\beta_b^{\mathcal{B}_{n-1}}$ of β with respect to $\mathcal{B}_{n-1}$ at b is the measure on B given by

$$\beta_b^{\mathcal{B}_{n-1}} = \delta_{b_1} \otimes \cdots \otimes \delta_{b_{n-1}} \otimes \mu^{\otimes \mathbb{N}}$$

It follows that for β-almost every $b \in B$ we have

$$\mathbb{E}(X_n | \mathcal{B}_{n-1})(b) = \int_B f(c_n, c_{n-1}^1 w) \mathrm{d}\beta_b^{\mathcal{B}_{n-1}}\,(c) = \int_G f(c_n, b_{n-1}^1 w) \mathrm{d}\mu\,(c_n) > L_0,$$

where the last inequality follows from assumption (2). Hence (6.7) holds and Z_n is a submartingale as claimed.

It is a classical fact (see [Wil91, section 10.9]) that the process $Z_{\min\{n,\rho\}}$ is also a submartingale with respect to $\mathcal{B}_n$. Hence, it satisfies the inequality

$$\int_B Z_1 \mathrm{d}\beta \leq \liminf_{n \to \infty} \int_B Z_{\min\{n,\rho(b)\}}(b) \mathrm{d}\beta\,(b)\,. \tag{6.8}$$

Since ρ is almost surely finite $\lim_{n \to \infty} Z_{\min\{n,\rho\}} = Z_\rho$ almost surely. Next we claim that $Z_{\min\{n,\rho\}}$ is bounded by an integrable function. Note that

$$|Z_{\min\{n,\rho\}}| \leq \sum_{i=1}^{\infty} \mathbf{1}_{\{b \in B : \rho(b) \geq i\}} |X_i - L_0|$$

and hence using (1) and the monotone convergence theorem

$$\begin{aligned}|Z_{\min\{n,\rho\}}| &\leq \sum_{i=1}^{\infty}\int_{\{b\in B:\rho(b)\geq i\}}|X_i - L_0|\mathrm{d}\beta \\ &\leq \sum_{i=1}^{\infty}(J_0+L_0)\beta(\{b\in B:\rho(b)\geq i\}) = (J_0+L_0)\int_B \rho\mathrm{d}\beta.\end{aligned}$$

Thus, using assumption (1), (6.8) and the dominated convergence theorem we obtain

$$0 < \int_B f(b_1,w)\mathrm{d}\beta\,(b) - L_0 = \int_B Z_1\mathrm{d}\beta \leq \lim_{n\to\infty}\int_B Z_{\min\{n,\rho(b)\}}(b)\mathrm{d}\beta\,(b) = \int_B Z_\rho\mathrm{d}\beta.$$

But from the definition of Z_ρ we have

$$\int_B Z_\rho\mathrm{d}\beta = \int_B \sum_{i=1}^{\rho(b)}(X_i(b)-L_0)\mathrm{d}\beta\,(b) = \int_B\sum_{i=1}^{\rho(b)}X_i(b)\mathrm{d}\beta\,(b) - L_0\int_B\rho\mathrm{d}\beta.$$

Putting the last two inequalities together yields the right inequality in (6.6) which finishes the proof. □

Similarly to the scheme leading to the construction of the contracted function $u_X^\kappa \in \mathrm{CH}_\mu(X)$ in Proposition 5.6, a key point in building the functions v_M which will participate in an application of Proposition 6.4 is showing that a certain family of functions $\mathcal{F}$ is uniformly (δ_0, I_0, L_0)-expanded by η according to Definition 5.2 for a certain choice of η. Let

$$\mathcal{D} := \{(p,v) : p\in\mathrm{Gr}_2(\mathbb{R}^3), v\in\mathfrak{g}\smallsetminus\mathfrak{r}_p\}. \tag{6.9}$$

For $(p,v)\in\mathcal{D}$ we choose $u_p\in\wedge^3\mathfrak{r}_p\smallsetminus\{0\}$ and define the multiplicative cocycle

$$f_{p,v}(g) := \frac{\|g(v\wedge u_p)\|}{\|v\wedge u_p\|}\Big/\frac{\|gu_p\|}{\|u_p\|}. \tag{6.10}$$

Note that the definition of $f_{p,v}$ is independent of the choice of u_p. We set

$$\mathcal{F}' := \{f_{p,v}\}_{(p,v)\in\mathcal{D}}.$$

Proposition 6.7. *Suppose that* ***S1*** *and* ***S2*** *hold. Then, for all large enough M and $y\in Y_M$, the family $\mathcal{F}'$ is uniformly (δ_0, I_0, L_0)-expanded by $\mu_{M,y}$. The parameters (δ_0, I_0, L_0) may depend on M but not on y.*

Proof. Take M large enough so that Proposition 6.5 holds for the submultiplicative function $N(g) := \|g\|_{\wedge^4\mathfrak{g}}\|g^{-1}\|_{\wedge^3\mathfrak{g}}$. Since $\sup_{f\in\mathcal{F}'}f(g)\leq N(g)$ it follows that there exists $I_0>0$ and $\delta_0>0$ such that

$$\int_G \sup_{f\in\mathcal{F}'} f^{\delta_0}(g)\mathrm{d}\mu_{M,y}\,(g)\leq I_0$$

for all $y\in Y_M$. This verifies condition (1) of Definition 5.2.

To verify condition (2) of Definition 5.2 we argue as follows. Assume M is large enough so that Proposition 6.6 is applicable and consider the additive cocycle $G\times\mathcal{D}\to\mathbb{R}$ given by $(g,p,v)\mapsto\log f_{p,v}(g)$. Condition (1) of Proposition 6.6 is satisfied with some J_0 because $\operatorname{supp}\mu$ is compact and condition (2) of Proposition 6.6 is satisfied with L_0 as in (5.2) by the discussion in §5.1. As an outcome we deduce equation (6.4) which reads as

$$\inf_{y\in Y_M}\inf_{(p,v)\in\mathcal{D}}\int\log f_{p,v}(g)\mathrm{d}\mu_{M,y}\,(g)\geq L_0.$$

This is exactly condition (2) of Definition 5.2. □

As an immediate corollary of Proposition 6.7 and Lemma 5.3 we have the following.

Corollary 6.8. *Suppose that* ***S1*** *and* ***S2*** *hold. Then, for all large enough M there exists $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$ there exists $0 < c = c(\delta) < 1$ such that for all $(p, v) \in \mathcal{D}$, $y \in Y_M$ and $u_p \in \wedge^3 \mathfrak{r}_p \smallsetminus \{0\}$ one has*

$$\int_G \left(\frac{\|g(v \wedge u_p)\|}{\|gu_p\|} \right)^{-\delta} \mathrm{d}\mu_{M,y}(g) \leq c \left(\frac{\|(v \wedge u_p)\|}{\|u_p\|} \right)^{-\delta}. \tag{6.11}$$

6.3. **Proof of non-atomicity of the limit measures.** We now apply the results of §6.2 to the space $Y = X \times X$. We fix once and for all $\kappa > 0$ such that $u_X^\kappa \in \mathrm{CH}_\mu(X)$ as in Proposition 5.6. We then take

$$u_{X\times X}(x, y) := u_X^\kappa(x) + u_X^\kappa(y),$$

so that $u_{X\times X} \in \mathrm{CH}_\mu(X \times X)$. We use the notions and notation of §6.2 for the G-action on $X \times X$ and the contracted function $u_{X\times X}$. In particular, we have the sublevel sets $(X \times X)_M$, the first return times $\rho_{M,(x,y)}$, the transition probability measures $\mu_{M,(x,y)}$ and the Markov operator $\mathrm{A}_{M,\mu}$.

Equipped with the above theory we are finally in a position to prove the following proposition which shows that one can apply Proposition 6.4 in our setting.

Proposition 6.9. *For all large enough $M > 0$ there exists a continuous proper function $v_M : (X \times X)_M \smallsetminus \Delta_X \to [0, \infty)$ such that*

$$v_M \in \mathrm{CH}_{\mathrm{A}_{M,\mu}}((X \times X)_M \smallsetminus \Delta_X).$$

Proof. Consider the submultiplicative function

$$R(g) := \max\{\|g^{\pm 1}\|_{\mathfrak{g}}, \|g^{\pm 1}\|_{\wedge^4 \mathfrak{g}} \|g^{\mp 1}\|_{\wedge^3 \mathfrak{g}}\}.$$

We choose M large enough and δ small enough so that Corollary 6.8 holds and Proposition 6.5 yields

$$\sup_{(x,y)\in(X\times X)_M} \int_G R(g)^{2\delta} \mathrm{d}\mu_{M,(x,y)}(g) =: J < \infty. \tag{6.12}$$

We let $c < 1$ be the constant satisfying (6.11). Define

$$v_M(x, y) := \mathrm{d}_X(x, y)^{-\delta}$$

and view it as a function on $(X \times X)_M \smallsetminus \Delta_X$. Here d_X is the metric discussed in §6.1. It is then clear that it is continuous and proper. We are left to prove the existence of constants $c' < 1$ and b' such that for all $(x, y) \in (X \times X)_M$,

$$\mathrm{A}_{M,\mu} v_M(x, y) = \int_G v_M(gx, gy) \mathrm{d}\mu_{M,(x,y)}(g) \leq c' v_M(x, y) + b'.$$

We will estimate the integral by splitting the domain of integration G into a compact piece and its complement and treat each piece separately. This splitting is done using the submultiplicative function R introduced above.

For any $T > 0$ let $G^{\leq T} := \{g \in G : R(g) \leq T\}$ and $G^{>T} := G \smallsetminus G^{\leq T}$. Then we have

$$\mathrm{A}_{M,\mu} v_M(x, y) = I_1 + I_2, \tag{6.13}$$

where,

$$I_1 := \int_{G^{\leq T}} v_M(gx, gy) \mathrm{d}\mu_{M,(x,y)}(g) \quad \text{and} \quad I_2 := \int_{G^{>T}} v_M(gx, gy) \mathrm{d}\mu_{M,(x,y)}(g).$$

We first estimate I_1. Fix a large T and a constant $c_1 < 1$ and apply Lemma 6.1 to the compact set $G^{\leq T}$ and the constant c_1 to obtain a neighbourhood of the diagonal $\mathcal{U} = \mathcal{U}_{T,c_1}$

satisfying the conclusion of Lemma 6.1. The integral I_1 is bounded separately according to whether $(x,y)\in\mathcal{U}$ or not:
Case 1. Assume $(x,y)\in(X\times X)_M\smallsetminus\mathcal{U}$. By compactness

$$\theta:=\inf\{\mathrm{d}_X(gx',gy'):(x',y')\in(X\times X)_M\smallsetminus\mathcal{U},\ g\in G^{\leq T}\}>0.$$

Hence in this case we have

$$I_1=\int_{G^{\leq T}}\mathrm{d}_X(gx,gy)^{-\delta}\mathrm{d}\mu_{M,(x,y)}(g)\leq\theta^{-\delta}.$$

Case 2. Assume $(x,y)\in(X\times X)_M\cap\mathcal{U}$. Let $u\in\wedge^3\mathfrak{r}_{\pi(x)}\smallsetminus\{0\}$. Then, by applying parts (2) and (3) of Lemma 6.1 and using Corollary 6.8 we get

$$\begin{aligned}I_1&=\int_{G^{\leq T}}\mathrm{d}_X(gx,gy)^{-\delta}\mathrm{d}\mu_{M,(x,y)}(g)\\&\leq c_1^{-2\delta}\int_{G^{\leq T}}\left(\frac{\|g(o_{x,y}\wedge u)\|}{\|gu\|}\right)^{-\delta}\mathrm{d}\mu_{M,(x,y)}(g)\\&\leq c_1^{-2\delta}c\left(\frac{\|(o_{x,y}\wedge u)\|}{\|u\|}\right)^{-\delta}\leq c_1^{-4\delta}c\,\mathrm{d}_X(x,y)^{-\delta}.\end{aligned}$$

In total, combining the two cases we arrive at the bound

$$I_1\leq c_1^{-4\delta}cv_M(x,y)+\theta^{-\delta}.$$

We now bound I_2. Notice that $R(g)=R(g^{-1})$ and by (6.1), for all $x,y\in X$ and $g\in G$ one has $\mathrm{d}_X(gx,gy)\leq R(g)\,\mathrm{d}_X(x,y)$. It follows that $\mathrm{d}_X(gx,gy)^{-\delta}\leq R(g)^{\delta}\,\mathrm{d}_X(x,y)^{-\delta}$. Therefore by (6.12) we have that

$$\begin{aligned}I_2&=\int_{G^{>T}}v_M(gx,gy)\mathrm{d}\mu_{M,(x,y)}(g)\\&\leq v_M(x,y)\int_{G^{>T}}R(g)^{\delta}\mathrm{d}\mu_{M,(x,y)}(g)\\&\leq v_M(x,y)T^{-\delta}\int_{G^{>T}}R(g)^{2\delta}\mathrm{d}\mu_{M,(x,y)}(g)\leq T^{-\delta}Jv_M(x,y).\end{aligned}$$

Combining the estimates for I_1 and I_2 we conclude that for all $(x,y)\in(X\times X)_M$ one has

$$\mathrm{A}_{M,\mu}\,v_M(x,y)\leq(c_1^{-4\delta}c+T^{-\delta}J)v_M(x,y)+\theta^{-\delta}.$$

If we choose c_1 close enough to 1 and T large enough then the constant $c':=c_1^{-4\delta}c+T^{-\delta}J$ is strictly less than 1, which completes the proof. Note that θ depends on T and c_1 (which determine $\mathcal{U}$) but that does not matter. □

Finally we conclude the proof of Theorem 2.1(b).

Proof of Theorem 2.1(b). Notice that since $\pi_*\nu=\bar{\nu}_{\mathrm{Gr}_2(\mathbb{R}^3)}$ is the Furstenberg measure of μ on $\mathrm{Gr}_2(\mathbb{R}^3)$ which is non-atomic (see Remark 1.4), ν is non-atomic as well. By Proposition 6.9 we may apply the criterion for the non-atomicity of the limit measures given in Proposition 6.4 which concludes the proof. □

Appendix A. Diophantine approximation and the geometry of numbers

In the past decades Furstenberg has been promoting an approach to attack a famous open problem in Diophantine approximation: *Are cubic numbers well approximable?* Recall that a number $\alpha \in \mathbb{R}$ is well approximable if the coefficients a_i in its continued fraction expansion $\alpha = [a_0; a_1, a_2, \ldots]$ form an unbounded sequence of integers. Lagrange's theorem asserts that α is a quadratic irrational number if and only if its continued fraction expansion is eventually periodic and hence clearly not well approximable. This could be proved by translating the problem into a dynamical problem about the action of the diagonal group $a_t = \operatorname{diag}\left(e^t, e^{-t}\right)$ acting on the space of lattices in the plane $\mathrm{PGL}_2(\mathbb{R}) / \mathrm{PGL}_2(\mathbb{Z})$. Furstenberg's approach says that the dynamical system $a_t \curvearrowright \mathrm{PGL}_2(\mathbb{R}) / \mathrm{PGL}_2(\mathbb{Z})$ which is tailored to detect quadratics, should be replaced with a dynamical system which is tailored to detect cubic irrationals. He then suggests the following characterisation of well approximability in terms of the dynamics on the space of 2-lattices X discussed in this paper.

Theorem A.1 (Furstenberg - unpublished)**.** *Let A denote the group of diagonal matrices in G and let $x = [\Lambda] \in X$ be the homothety class of the 2-lattice $\Lambda = \operatorname{span}_{\mathbb{Z}}\{v, w\}$ spanned by $v, w \in \mathbb{R}^3 \smallsetminus \{0\}$. Assume that $\Lambda \cap p = \{0\}$ for $p = \operatorname{span}_{\mathbb{R}}\{e_i, e_j\}$ any one of the three planes fixed by A. Then the orbit Ax is unbounded in X if and only if one of the ratios v_i/w_i is well approximable, for $i = 1, 2, 3$.*

Before we proceed to show that the dynamical system $A \curvearrowright X$ can detect cubic numbers in a certain sense, we introduce a notion in *geometry of numbers* that will be important for our discussion. Consider a lattice $L \subset \mathbb{R}^3$ and an L-rational line W (where W is L-rational if $L \cap W \neq \{0\}$). We define the *directional* 2*-lattice*

$$L_W := \pi^W(L)$$

where π^W denotes the orthogonal projection onto $W^\perp$. The term comes from visualising L_W as representing what L looks like when one is looking in the direction of W. We set

$$\mathfrak{D}(L) = \{[L_W] : W \in \mathbb{P}\mathbb{R}^3 \text{ is } L\text{-rational}\} \subset X.$$

We now wish to describe subcollections of $\mathfrak{D}(L)$ which are obtained by conditioning on W. Note that an L-rational line $W \in \mathbb{P}\mathbb{R}^3$ is characterised by the generator v_W (well defined up to sign) of $L \cap W$. Given a subset $S \subset \mathbb{R}^3$ we define the set of conditioned directional lattices defined by L and S to be

$$\mathfrak{D}_S(L) = \{[L_W] : W \in \mathbb{P}\mathbb{R}^3 \text{ is } L\text{-rational and } v_W \in S\} \subset X.$$

Sometimes, instead of writing L_W we write L_v, where $v = v_W$.

Consider a lattice $L \subset \mathbb{R}^3$ which is obtained in the following manner. Let $\mathbb{K}$ be a totally real number field of degree 3 over $\mathbb{Q}$ and for $i = 1, 2, 3$ let σ_i be its distinct embeddings into the reals. Let $\varphi : \mathbb{K} \to \mathbb{R}^3$ be the so-called geometric embedding given by $\varphi(\alpha) := (\sigma_i(\alpha))_1^3$. It is well known that if $\mathcal{O}_{\mathbb{K}}$ denotes the ring of integers in $\mathbb{K}$ then $L := \varphi(\mathcal{O}_{\mathbb{K}})$ is a lattice in $\mathbb{R}^3$. Let $\mathrm{N} : \mathbb{R}^3 \to \mathbb{R}$ denote the cubic form given by $\mathrm{N}(v) = v_1 v_2 v_3$, so that for $\alpha \in \mathbb{K}$ one has $N_{\mathbb{K}/\mathbb{Q}}(\alpha) = \mathrm{N}(\varphi(\alpha))$.

The lattice L has a very special relationship with the surface

$$S := \{v \in \mathbb{R}^3 : \mathrm{N}(v) = \pm 1\}.$$

Namely,

$$S \cap L = S \cap L_{\text{prim}} = \{\varphi(\alpha) : \alpha \in \mathcal{O}_{\mathbb{K}}^\times\}^{4}.$$

[4] L_{prim} denotes the collection of primitive vectors in L.

In particular, the group

$$A_L := \{\operatorname{diag}(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha)) : \alpha \in \mathcal{O}_{\mathbb{K}}^{\times}\}, \tag{A.1}$$

which has a finite index subgroup which is a lattice in $A \simeq \mathbb{R}^2$ by Dirichlet's unit theorem, acts transitively and simply on $S \cap L$. This furnishes the link between Furstenberg's criterion for well approximability and cubic numbers.

Corollary A.2. *Let $\mathbb{K}$, $L = \varphi(\mathcal{O}_{\mathbb{K}})$ and $S = \{v \in \mathbb{R}^3 : \mathrm{N}(v) = \pm 1\}$ be as above. Then the collection of conditioned directional lattices $\mathfrak{D}_S(L)$ is unbounded in X if and only if for some $1 \le i \le 3$, $\sigma_i(\mathbb{K}) \smallsetminus \mathbb{Q}$ is composed of well approximable numbers.*

Proof. For $v \in \mathbb{R}^3$ we denote by $\pi^v : \mathbb{R}^3 \to v^{\perp}$ the orthogonal projection with kernel $\mathbb{R}v$. Let $\mathbf{1} = (1,1,1) \in L$ and denote $\Lambda = \pi^{\mathbf{1}}(L)$. Since the group in (A.1) contains a finite index subgroup which is cocompact in A, the unboundedness of $A[\Lambda]$ is equivalent to the unboundedness of $A_L[\Lambda]$. But

$$\begin{aligned} A_L[\Lambda] &= \{\operatorname{diag}(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha))\,([\pi^{\mathbf{1}}(L)]) : \alpha \in \mathcal{O}_{\mathbb{K}}^{\times}\} \\ &= \{[\pi^{\varphi(\alpha)}(L)] : \alpha \in \mathcal{O}_{\mathbb{K}}^{\times}\} = \{[\pi^{v}(L)] : v \in L \cap S\} = \mathfrak{D}_S(L), \end{aligned}$$

where we have used the fact that for $g \in G$ and $v \in L$, $g\pi^v(L) = \pi^{gv}(g^{-t}L)$ and the fact that for $a \in A_L$, we have that $a^{-t}L = L$.

Let $\alpha, \beta \in \mathcal{O}_{\mathbb{K}}$ be such that $\{1, \alpha, \beta\}$ forms a basis of $\mathcal{O}_{\mathbb{K}}$ over $\mathbb{Q}$. Denote $\alpha' = \alpha - \frac{1}{3}\operatorname{Tr}(\alpha)$ and similarly denote $\beta' = \beta - \frac{1}{3}\operatorname{Tr}(\beta)$. It follows that Λ is spanned by $\varphi(\alpha')$ and $\varphi(\beta')$. Hence, by Furstenberg's criterion (Theorem A.1) we deduce that $\mathfrak{D}_S(L)$ is unbounded if and only if there exists $1 \le i \le 3$ such that the ratio $\sigma_i(\alpha'/\beta')$ is well approximable. Now it is not hard to see that for a given cubic real field $\mathbb{F}$, since $\mathbb{F} \smallsetminus \mathbb{Q}$ is a single $\mathrm{GL}_2(\mathbb{Q})$-orbit (under the action by Möbius transformations), and since this action preserves well approximability, then either all elements of $\mathbb{F} \smallsetminus \mathbb{Q}$ are well approximable or non of them are. This concludes the proof of the Corollary. □

Let

$$\Delta := \{p \in \mathrm{Gr}_2(\mathbb{R}^3) : p \text{ contains one of the axis } \mathbb{R}e_i,\ i = 1,2,3\}$$

be the projective triangle defined by the axis. Recasting in terms of conditioned directional lattices the common belief that real cubic numbers should be well approximable and even generic for the Gauss map we propose the following.

Conjecture A.3. *Let L and S be as in Corollary A.2. Then the closure in X of $\mathfrak{D}_S(L)$ contains $\pi^{-1}(\Delta)$.*

In the introduction we stated Conjecture 1.1 which follows from the following conjecture. We view it as an analogue to Conjecture A.3. We use the notation $V_Q^1 = \{v : Q(v) = 1\}$ for $Q(v) := 2v_1v_3 - v_2^2$ from the introduction and also $\mathcal{C} \subset \mathrm{Gr}_2(\mathbb{R}^3)$ for the circle of isotropic subspaces defined before Theorem 1.10.

Conjecture A.4. *The closure of $\mathfrak{D}_{V_Q^1}(\mathbb{Z}^3)$ contains $\pi^{-1}(\mathcal{C})$.*

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