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Equidistribution of divergent orbits and continued fraction expansion of rationals

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Abstract

We establish an equidistribution result for pushforwards of certain locally finite algebraic measures in the adelic extension of the space of lattices in the plane. As an application of our analysis, we obtain new results regarding the asymptotic normality of the continued fraction expansions of most rationals with a high denominator as well as an estimate on the length of their continued fraction expansions.

By similar methods, we also establish a complementary result to Zaremba's conjecture. Namely, we show that given a bound M, for any large q, the number of rationals $p/q \in [0,1]$ for which the coefficients of the continued fraction expansion of p/q are bounded by M is $o(q^{1-\epsilon})$ for some $\epsilon > 0$, which depends on M.

1. *Introduction*

1.1. *Continued fraction expansion of rationals*

We begin by describing the main application of our results. Let $T : (0,1] \to [0,1]$ denote the Gauss map $T(s) := \left\{s^{-1}\right\} := s^{-1} - \lfloor s^{-1} \rfloor$. Let $\nu_{\text{Gauss}} = ((1+s)\ln 2)^{-1} ds$ denote the Gauss–Kuzmin measure on [0,1]. A number $s \in (0,1]$ is rational if and only if $T^i(s) = 0$ for some i (in which case $T^{i+1}(s)$ is not defined). In this case, we denote this i by $\mathrm{len}(s)$ which is the length of the (finite) continued fraction expansion (c.f.e.) of s. We also set

$$\nu_s = \frac{1}{\mathrm{len}(s)} \sum_{i=0}^{\mathrm{len}(s)-1} \delta_{T^i(s)}.$$

Throughout we abuse notation and denote

$$(\mathbb{Z}/q\mathbb{Z})^\times = \{1 \leqslant p \leqslant q : \gcd(p,q) = 1\}.$$

Theorem 1.1. *There exist sets $W_q \subseteq (\mathbb{Z}/q\mathbb{Z})^\times$ with $\lim_{q\to\infty} \frac{|W_q|}{\varphi(q)} = 1$, such that for any choice of $p_q \in W_q$, we have that*

(1) $\frac{\mathrm{len}(p_q/q)}{2\ln(q)} \to \frac{\ln(2)}{\zeta(2)}$ *where ζ is the Riemann zeta function.*

(2) $\nu_{p_q/q} \xrightarrow{\mathrm{w}^*} \nu_{\text{Gauss}}$.

Remark 1.2. Let $\mathbf{w}$ be a finite word on $\mathbb{N}$. It is well known, and indeed follows from the ergodicity of T with respect to ν_{Gauss}, that for Lebesgue almost any x, the asymptotic frequency of appearances of $\mathbf{w}$ in the c.f.e. of x equals

$$\nu_{\text{Gauss}}(\mathbf{w}) \stackrel{\text{def}}{=} \nu_{\text{Gauss}}\left(\{y \in [0,1] : \text{ the c.f.e. of } y \text{ starts with } \mathbf{w}\}\right). \tag{1.1}$$

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Let us denote by $\nu_{p/q}(\mathbf{w})$ the *frequency* of the word $\mathbf{w}$ in the c.f.e. of p/q; that is, the number of appearances of $\mathbf{w}$ in the c.f.e. of p/q divided by $\mathrm{len}(p/q)$. Then, it is easy to see that since the endpoints of the interval given by the set in (1.1) have zero ν_{Gauss} measure, then the weak* convergence in part (2) of Theorem 1.1 implies that for any finite word $\mathbf{w}$ over $\mathbb{N}$, we have that $\nu_{p_q/q}(\mathbf{w}) \to \nu_{\mathrm{Gauss}}(\mathbf{w})$.

An obvious corollary of Theorem 1.1 (together with the fact that $\mathrm{len}(p/q) \leqslant 2\log_2(q)$) is obtained by averaging over $p \in (\mathbb{Z}/q\mathbb{Z})^\times$ as follows.

Corollary 1.3. (1) *Let* $\bar{\nu}_q = \varphi(q)^{-1} \sum_{p\in(\mathbb{Z}/q\mathbb{Z})^\times} \nu_{p/q}$. *Then* $\bar{\nu}_q \xrightarrow{\mathrm{w}^*} \nu_{\mathrm{Gauss}}$.

(2) *Let* $\overline{\mathrm{len}}(q) = \varphi(q)^{-1} \sum_{p\in(\mathbb{Z}/q\mathbb{Z})^\times} \mathrm{len}(p/q)$. *Then* $\frac{\overline{\mathrm{len}}(q)}{2\ln q} \to \frac{\ln 2}{\zeta(2)}$.

This corollary was first obtain by Heilbronn [8] who also computed an error term, which was later improved by Ustinov [15]. The upgrade from Corollary 1.3 to Theorem 1.1 is almost automatic when the discussion is lifted to the space of lattices as can be seen in § 2.4. It seems not to be available when the discussion stays in the classical realm of the Gauss map. Running over all $1 \leqslant p \leqslant q$ and not just $(p,q) = 1$, Bykovskii [2] showed that $\frac{1}{q}\sum_1^q(\mathrm{len}(\frac{p}{q}) - \frac{2\ln(2)}{\zeta(2)}\ln q)^2 \ll \ln q$.

We also note that averaged versions of Theorem 1.1 with an extra average over q were obtained by Dixon [3] who showed that for any $\varepsilon > 0$, there exists $c > 0$ such that

$$\#\left\{(p,q) : \begin{array}{l} 1 \leqslant p \leqslant q \leqslant x, \\ \left|\frac{\mathrm{len}(p/q)}{2\ln(2)} - \frac{\ln q}{\zeta(2)}\right| < \frac{1}{2}(\ln q)^{-\frac{1}{2}+\varepsilon} \end{array}\right\} \leqslant x^2 \exp\left(-c\ln^{\varepsilon/2}(x)\right),$$

which was later improved by Hensley in [9]. See also [1] and [16] for construction of normal numbers with respect to c.f.e. using rational numbers.

1.2. *Contrast to Zaremba's conjecture*

Recall that Zaremba's conjecture [18] asserts that there exists $M > 0$ such that for all q, there exists $p \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that all the coefficients in the c.f.e. of p/q are bounded by M. Theorem 1.1 may be interpreted as saying that Zaremba is looking for a needle in a haystack. In fact, while Theorem 1.1 asserts that the set of p/q which are good for Zaremba is of size $o(q)$, the following strengthening says that it is actually $o(q^{1-\epsilon})$.

Theorem 1.4. *For each M, there exists $\epsilon > 0$ such that*

$$\#\left\{p \in (\mathbb{Z}/q\mathbb{Z})^\times : \text{the coefficients of the c.f.e. of } p/q \text{ are bounded by } M\right\} = o(q^{1-\epsilon}).$$

As noted to us by Moshchevitin, in [10], a stronger statement is achieved where ϵ is given explicitly.

1.3. *Divergent geodesics*

Let $G = \mathrm{PGL}_2(\mathbb{R})$, $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$ and $X_2 = \Gamma\backslash G$. The space X_2 is naturally identified with the space of homothety classes of lattices in the plane where the coset Γg corresponds to the (homothety class of the) lattice $\mathbb{Z}^2 g$. We shall refer to $\mathbb{Z}^2$ as the standard lattice and denote its class in X_2 by x_0. We let G and its subgroups act on X_2 from the right and usually abuse notation and write elements of G as matrices. Consider the subgroups of G,

$$A = \left\{a(t) = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2}\end{pmatrix} : t \in \mathbb{R}\right\}; \quad U = \left\{u_s = \begin{pmatrix} 1 & s \\ 0 & 1\end{pmatrix} : t \in \mathbb{R}\right\}. \tag{1.2}$$

Theorem 1.1 is a consequence of a certain equidistribution theorem regarding collections of divergent orbits of the diagonal group which we now wish to discuss. It is not hard to see that if $s = p/q$ is a rational in reduced form, then the A-orbit $x_0 u_s A$ is divergent; that is, the map $t \mapsto x_0 u_s a(t)$ is a proper embedding of $\mathbb{R}$ in X_2. In fact, for $t < 0$, this lattice contains the vector $e^{t/2}(0,1)$ which is of length $e^{t/2} \to 0$ as $t \to -\infty$ and for $t > 0$, the lattice contains the vector $(q,-p)u_s a(t) = (qe^{-t/2}, 0)$ which is of length $\leqslant 1$ when $t \geqslant 2\ln q$ and goes to zero as $t \to \infty$. So, the interesting life span of the orbit $x_0 u_s A$ is the interval $\{x_0 u_s a(t) : t \in [0, 2\ln q]\}$. We therefore define for $p \in (\mathbb{Z}/q\mathbb{Z})^\times$,

$$\delta_{x_0 u_{p/q}}^{[0,2\ln q]} = \frac{1}{2\ln q}\int_0^{2\ln q} \delta_{x_0 u_{p/q} a(t)} \mathrm{dt}, \tag{1.3}$$

which means that for a bounded continuous function on X_2 we have

$$\int_{X_2} f d\delta_{x_0 u_{p/q}}^{[0,2\ln q]} := \frac{1}{2\ln q}\int_0^{2\ln q} f(x_0 u_{p/q} a(t)) \mathrm{dt}.$$

Finally, let μ_{Haar} denote the unique G-invariant probability measure on X_2. The tight relation between the A-action on X_2 and continued fractions is well understood. Indeed, we deduce Theorem 1.1 from results in the space X_2 which we now describe.

THEOREM 1.5. *As $q \to \infty$, we have that*

$$\frac{1}{\varphi(q)} \sum_{p \in (\mathbb{Z}/q\mathbb{Z})^\times} \delta_{x_0 u_{p/q}}^{[0,2\ln q]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}.$$

COROLLARY 1.6. *There exist sets $W_q \subseteq (\mathbb{Z}/q\mathbb{Z})^\times$ with $\lim_{q\to\infty} \frac{|W_q|}{\varphi(q)} = 1$, such that for any choice of $p_q \in W_q$, we have that $\delta_{x_0 u_{p/q}}^{[0,2\ln(q)]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$.*

As mentioned before, although it seems stronger, Corollary 1.6 follows from Theorem 1.5 using only the fact that μ_{Haar} is A-ergodic. See § 2.4 for details.

We will prove Theorem 1.5 as a consequence of the following more general equidistribution result. We say that a sequence of probability measures η_n does not exhibit escape of mass if any weak* accumulation point of it is a probability measure.

THEOREM 1.7. *Let $\Lambda_q \subset (\mathbb{Z}/q\mathbb{Z})^\times$ be subsets such that*

(i) $\lim \frac{\ln|\Lambda_q|}{\ln q} = 1$,

(ii) *the sequence of measures $\frac{1}{|\Lambda_q|}\sum_{p\in\Lambda_q} \delta_{x_0 u_{p/q}}^{[0,2\ln q]}$ does not exhibit escape of mass.*

Then $\frac{1}{|\Lambda_q|}\sum_{p\in\Lambda_q} \delta_{x_0 u_{p/q}}^{[0,2\ln q]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$.

REMARK 1.8. Note that in Theorem 1.5, we have that $|\Lambda_q| = \varphi(q)$ is the Euler's totient function and it is well known that $\lim \frac{\ln\varphi(q)}{\ln q} = 1$, which is condition (i) above (indeed, this claim follows from the multiplicative nature of the totient function). Thus, in order to deduce Theorem 1.5 from Theorem 1.7, we only need to show that there is no escape of mass.

1.4. *A more conceptual viewpoint*

Let $X_n = \mathrm{PGL}_n(\mathbb{Z})\backslash \mathrm{PGL}_n(\mathbb{R})$ be identified with the space of homothety classes of lattices in $\mathbb{R}^n$ and let $A < \mathrm{PGL}_n(\mathbb{R})$ denote the connected component of the identity of the full diagonal group. It is well known (see [**14**]) that an orbit xA is divergent (that is, the map $a \mapsto xa$ from

A to X_n is proper), if and only if it contains a homothety class of an integral lattice. It is not hard to show that in this case, there is a unique such integral lattice which minimizes the covolume. Indeed, if Λ is an integral lattice and $\pi_i(\Lambda)$ is its projection onto the ith coordinate, then it must be $k_i\mathbb{Z}$ for some $k_i \in \mathbb{N}$. The covolume is minimized exactly when we move along the A-orbit to a lattice where $k_i = 1$ for all i. We refer to the square of this covolume as the *discriminant* of the divergent orbit. Let $\mathcal{H}_q(n)$ be the finite collection of sublattices of $\mathbb{Z}^n$ of covolume q having the property that $\pi_i(\Lambda) = \mathbb{Z}$ for $i = 1, \ldots, n$. We leave it as an exercise to show that the collection of divergent orbits of discriminant q^2 is exactly $\{xA : x \in \mathcal{H}_q(n)\}$. By abuse of notation, we also think about $\mathcal{H}_q(n)$ as a subset of X_n. In dimension 2, we have $\mathcal{H}_q(2) = \{\mathbb{Z}^2\left(\begin{smallmatrix}1 & p\\ 0 & q\end{smallmatrix}\right) : p \in (\mathbb{Z}/q\mathbb{Z})^\times\}$. Note that the collection of orbits $\{x_0 u_{p/q} A : p \in (\mathbb{Z}/q\mathbb{Z})^\times\}$ in X_2 is the same as $\{xA : x \in \mathcal{H}_q(2)\}$.

In Theorem 1.5, we truncated the divergent orbits $\{xA : x \in \mathcal{H}_q(2)\}$, since we wanted to use the weak* topology which is defined on the space of *finite* measures on X_2. It is conceptually better to present a certain topology on the space of *locally finite* measures which will allow Theorem 1.5 to be restated and conveniently generalized to a convergence statement involving the natural locally finite A-invariant measures supported on the collection of divergent orbits $\{xA : x \in \mathcal{H}_q(2)\}$. To this end, let us denote by μ_{xA} the measure on X_2 obtained by pushing a fixed choice of Haar measure on A via the map $a \mapsto xa$ (where xA is divergent and hence the map is proper so that the pushed measure is indeed locally finite). In dimension 2, we identify $A \simeq \mathbb{R}$ by $t \mapsto a(t)$ and choose the standard Lebesgue measure coming from this identification.

Let Z be a locally compact second countable Hausdorff space and let $\mathcal{M}(Z)$ denote the space of locally finite-positive Borel measures on Z and let $\mathbb{P}\mathcal{M}(Z)$ denote the space of homothety classes of such (non-zero) measures. For $\mu \in \mathcal{M}(Z)$, we let $[\mu]$ denote its class. It is straightforward to define a topology on $\mathbb{P}\mathcal{M}(Z)$ such that the following are equivalents for $[\mu_n], [\mu] \in \mathbb{P}\mathcal{M}(Z)$ (see [**13**]).

(1) $\lim[\mu_n] = [\mu]$.

(2) There exist constants c_n such that for any compact set $K \subset Z$, $c_n\mu_n|_K \xrightarrow{\mathrm{w}^*} \mu|_K$ (which means that for every $f \in C_c(Z)$, $c_n \int f d\mu_n \to \int f d\mu$).

(3) For every $f, g \in C_c(Z)$ for which $\int g d\mu \neq 0$, $\lim_{n\to\infty} \frac{\int f d\mu_n}{\int g d\mu_n} \to \frac{\int f d\mu}{\int g d\mu}$ (and in particular, $\int g d\mu_n \neq 0$ for all large enough n).

It is straightforward to see that if c_n, c_n' are sequences of scalars such that $c_n\mu_n$ and $c_n'\mu_n$ both converge to μ in the sense of ((2)), then $c_n/c_n' \to 1$.

We propose the following.

Conjecture 1.9. For any dimension n, as $q \to \infty$, the homothety class of the locally finite measure $\sum_{x\in\mathcal{H}_q(n)} \mu_{xA}$ converges in the above topology to the homothety class of the $\mathrm{PGL}_n(\mathbb{R})$-invariant measure on X_n.

Theorem 1.10. *Conjecture 1.9 holds for $n = 2$.*

We will see in Lemma 3.10 that Theorem 1.10 follows from (and is, in fact, equivalent to) Theorem 1.5.

1.5. *Adelic orbits*

We now concentrate on the 2-dimensional case. Yet, another conceptual view point that we wish to present and which puts the statement of Theorem 1.10 in a natural perspective is as follows. Let $\mathbb{A}$ denote the ring of adeles over $\mathbb{Q}$ and consider the space $X_{\mathbb{A}} = \Gamma_{\mathbb{A}}\backslash G_{\mathbb{A}}$ (where $G_{\mathbb{A}} = \mathrm{PGL}_2(\mathbb{A})$ and $\Gamma_{\mathbb{A}} = \mathrm{PGL}_2(\mathbb{Q})$). Let $A_{\mathbb{A}} < G_{\mathbb{A}}$ denote the subgroup of diagonal matrices.

Note that the orbit $\tilde{x}_0 A_{\mathbb{A}}$ is a closed orbit (where $\tilde{x}_0$ denotes the identity coset $\Gamma_{\mathbb{A}}$). In particular, fixing once and for all a Haar measure on $A_{\mathbb{A}}$, we obtain a Haar measure on the quotient $\mathrm{stab}_{A_{\mathbb{A}}}(\tilde{x}_0)\backslash A_{\mathbb{A}}$ and by pushing the latter into $X_{\mathbb{A}}$ via the proper embedding induced by the map $a \mapsto \tilde{x}_0 a$, we obtain an $A_{\mathbb{A}}$-invariant locally finite measure $\mu_{\tilde{x}_0 A_{\mathbb{A}}}$ supported on the closed orbit $\tilde{x}_0 A_{\mathbb{A}}$. Theorem 1.10 (and hence Theorem 1.5) is implied (and, in fact, equivalent as will be seen by the proof) to the following.

THEOREM 1.11. *For any sequences $g_i \in G_{\mathbb{A}}$ such that* (i) *the real component of g_i is trivial,* (ii) *the projection of g_i to $G_{\mathbb{A}}/A_{\mathbb{A}}$ is unbounded, the sequence of homothety classes of the locally finite measures $(g_i)_*\mu_{\tilde{x}_0 A_{\mathbb{A}}}$ converges in the topology introduced above to the homothety class of the $G_{\mathbb{A}}$-invariant measure on $X_{\mathbb{A}}$.*

In fact, we propose the following.

CONJECTURE 1.12. In the statement of Theorem 1.11, one can omit requirement (i) from the sequence g_i.

The main result in [**11**] can be interpreted as saying that if $g_i \in \mathrm{PGL}_2(\mathbb{R})$ is unbounded modulo the diagonal group A, then the homothety class of $(g_i)_*\mu_{x_0 A}$ converges in the topology introduced above to the homothety class of μ_{Haar}. It seems plausible (although not immediate as far as we can see) that a proof of Conjecture 1.12 might be obtained by combining the techniques of [**11**] and ours.

1.6. *Structure of the paper and outline of the proofs*

In § 2, we prove Theorem 1.7. We show that any weak* accumulation point of the sequence of measures appearing in the statement (which is automatically A-invariant) has the same entropy with respect to say, $a(1)$, as the measure μ_{Haar}. Since μ_{Haar} is the unique measure with maximal entropy, this establishes that μ_{Haar} is the only possible weak* accumulation point of the above sequence and finishes the proof. We then deduce Theorem 1.5 by verifying that the two conditions for applying Theorem 1.7 hold for $\Lambda_q = (\mathbb{Z}/q\mathbb{Z})^\times$. Here, the non-trivial part is to show that in this case, there is no escape of mass.

In § 3, we prove that Theorems 1.5, 1.10 and 1.11 are equivalent. In § 4, we review the relation between the A action on X_2 and the Gauss map and isolate the necessary technical statements which will allow us to deduce Theorem 1.1 from Theorem 1.5. We end § 4 by proving Theorem 1.4 the proof of which follows along similar lines as the proof of Theorem 1.1.

2. *Proof of the main theorem*

In this section, we prove Theorem 1.7 and deduce Theorems 1.5. We start with some notation and definitions, and then, in § 2.1, make a minor reduction to replace the measures that appear in the statement of Theorem 1.7 with a discrete version of themselves which is better suited for the entropy argument. In § 2.2, we state the main tool we use in the proof (uniqueness of measure with maximal entropy) and establish maximal entropy of the appropriate weak* limits which finishes the proof of Theorem 1.7. In § 2.3, we verify that the measures appearing in the statement of Theorem 1.5 satisfy the conditions in Theorem 1.7 and by that conclude the proof of Theorem 1.5. Finally, in § 2.4, we use the ergodicity of the Haar measure in order to upgrade the averaged result from Theorem 1.5 to Corollary 1.6.

In this section, we set $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and are interested in equidistribution in the space $X = X_2 = \Gamma\backslash G \cong \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})$. The group G then acts naturally on X and on the space of functions on X. We denote the positive diagonal and upper unipotent subgroups of $\mathrm{SL}_2(\mathbb{R})$ by A, U, respectively, as in (1.2).

As mentioned in § 1.3, we will work with measures on partial A-orbit defined as follows.

DEFINITION 2.1. (i) For a finite set $\Lambda \subseteq X$, we write $\delta_\Lambda = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\delta_x$. We will sometimes write $\delta_{p/q}$ instead of $\delta_{x_0 u_{p/q}}$, and given a set $\Lambda_q \subseteq (\mathbb{Z}/q\mathbb{Z})^\times$, we will identify it with the set $\left\{x_0 u_{p/q} : p \in \Lambda_q\right\} \subseteq X$, and simply write δ_{Λ_q}.

(ii) Given a measure μ, a segment $[a,b] \subseteq \mathbb{R}$ and an integer $k \in \mathbb{Z}$, we define the averages $\mu^{[a,b]} = \frac{1}{b-a}\int_a^b a(-t)\mu \mathrm{dt}$ and $\mu^k = \frac{1}{k}\sum_0^{k-1} a(-j)\mu$. Note that with these definitions $\delta_x^k = \frac{1}{k}\sum_0^{k-1}\delta_{xa(j)}$ and similarly, $\delta_x^{[a,b]} = \frac{1}{b-a}\int_a^b \delta_{xa(t)}dt$.

2.1. *A reduction*

The following statement is very similar to that of Theorem 1.7. The only difference is that the continuous interval $[0, 2\ln q]$ is replaced by the discrete first half of it $\mathbb{Z} \cap [0, \ln q]$.

THEOREM 2.2. *Let $\Lambda_q \subset (\mathbb{Z}/q\mathbb{Z})^\times$ be subsets such that*

(i) $\lim \frac{\ln|\Lambda_q|}{\ln q} = 1$,

(ii) *the sequence of measures $\delta_{\Lambda_q}^{\lfloor \ln q\rfloor}$ does not exhibit escape of mass (that is, any weak* limit of it is a probability measure).*

Then $\delta_{\Lambda_q}^{\lfloor \ln q\rfloor} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$.

For entropy considerations, it will be more convenient to work with powers of a single transformation rather than with the continuous group A. As will be seen shortly, replacing $[0, 2\ln q]$ by its first half will also be more convenient. Thus, our plan is to establish Theorem 2.2 but first we deduce Theorem 1.7 from it.

Proof of Theorem 1.7 (given Theorem 2.2). Assume that Λ_q satisfies assumptions (i) and (ii) of Theorem 1.7. Let $\tau : X \to X$ be the automorphism taking a lattice to its dual and recall that if $x = \Gamma g$, then $\tau(x) = \Gamma(g^{-1})^{tr}$, where tr means the transpose, and hence $\tau(xa(t)) = \tau(x)a(-t)$ for all $t \in \mathbb{R}$. Let us also denote $p \mapsto p'$ the map from $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/q\mathbb{Z})^\times$ for which $pp' = -1$ modulo q. We claim that

$$\delta_{\Lambda_q}^{[\ln q, 2\ln q]} = \tau_* \delta_{\Lambda_q'}^{[0,\ln q]}. \tag{2.1}$$

To show (2.1), we first observe the following: Fix $p \in (\mathbb{Z}/q\mathbb{Z})^\times$ and let $q' \in \mathbb{Z}$ be such that $(-p)p' + qq' = 1$. We then have

$$\begin{aligned} x_0 u_{p/q} a(2\ln q) &= \Gamma\begin{pmatrix}1 & p/q\\ 0 & 1\end{pmatrix}\begin{pmatrix}q^{-1} & 0\\ 0 & q\end{pmatrix} = \Gamma\begin{pmatrix}q^{-1} & p\\ 0 & q\end{pmatrix}\\ &= \Gamma\begin{pmatrix}q & -p\\ -p' & q'\end{pmatrix}\begin{pmatrix}q^{-1} & p\\ 0 & q\end{pmatrix} = \Gamma\begin{pmatrix}1 & 0\\ -p'/q & 1\end{pmatrix} = \tau(x_0 u_{p'/q}). \end{aligned}$$

It now follows that for all t, $x_0 u_{p/q} a(2\ln q - t) = \tau(x_0 u_{p'/q} a(t))$, and hence (2.1) follows. We conclude from (2.1) that

$$\delta_{\Lambda_q}^{[0,2\ln q]} = \frac{1}{2}\delta_{\Lambda_q}^{[0,\ln q]} + \frac{1}{2}\tau_*\delta_{\Lambda_q'}^{[0,\ln q]}. \tag{2.2}$$

Since $\delta_{\Lambda_q}^{[0,2\ln q]}$ does not exhibit escape of mass, the same is true for the sequence $\delta_{\Lambda_q}^{[0,\ln q]}$ (as well as $\delta_{\Lambda'_q}^{[0,\ln q]}$). Since

$$\delta_{\Lambda_q}^{[0,\ln q]} = \frac{\lfloor \ln q \rfloor}{\ln q}\delta_{\Lambda_q}^{[0,\lfloor \ln q \rfloor]} + \left(1 - \frac{\lfloor \ln q \rfloor}{\ln q}\right)\delta_{\Lambda_q}^{[\lfloor \ln q \rfloor, \ln q]}, \tag{2.3}$$

and $\frac{\lfloor \ln q \rfloor}{\ln q} \to 1$, we conclude that the sequence $\delta_{\Lambda_q}^{[0,\lfloor \ln q \rfloor]}$ does not exhibit escape of mass. Finally, since

$$\delta_{\Lambda_q}^{[0,\lfloor \ln q \rfloor]} = \int_0^1 a(-t)_* \delta_{\Lambda_q}^{\lfloor \ln q \rfloor} dt, \tag{2.4}$$

we conclude that $\delta_{\Lambda_q}^{\lfloor \ln q \rfloor}$ does not exhibit escape of mass. We therefore obtain Λ_q satisfy conditions (i) and (ii) from Theorem 2.2 and since we assume the validity of this theorem at this point, we conclude that $\delta_{\Lambda_q}^{\lfloor \ln q \rfloor} \xrightarrow{\text{w}^*} \mu_{Haar}$. Since μ_{Haar} is $a(t)$-invariant, equation (2.4) implies $\delta_{\Lambda_q}^{[0,\lfloor \ln q \rfloor]} \xrightarrow{\text{w}^*} \mu_{Haar}$. In turn, by (2.3), we get that $\delta_{\Lambda_q}^{[0,\ln q]} \xrightarrow{\text{w}^*} \mu_{Haar}$.

A similar application of Theorem 2.2 for Λ'_q results in the conclusion that $\delta_{\Lambda'_q}^{[0,\ln q]} \xrightarrow{\text{w}^*} \mu_{Haar}$ and since μ_{Haar} is τ-invariant, we obtain from (2.2) that $\delta_{\Lambda_q}^{[0,2\ln q]} \xrightarrow{\text{w}^*} \mu_{Haar}$ as claimed. □

2.2. *Maximal entropy*

We briefly recall the notion of entropy mainly to set the notation. The reader is referred to any standard textbook on the subject for a more thorough account. See, for example, [**6**, **17**]. Recall that given a measurable space $(Y, \mathcal{B})$, a finite measurable partition $\mathcal{P}$ of Y and a probability measure μ on Y, we define the entropy of μ with respect to $\mathcal{P}$ to be

$$H_\mu(\mathcal{P}) = -\sum_{P_i \in \mathcal{P}} \mu(P_i) \ln(\mu(P_i)).$$

We refer to the sets composing the partition $\mathcal{P}$ as the *atoms* of $\mathcal{P}$. Given a μ-preserving transformation $T : Y \to Y$, we define

$$\forall k < \ell \in \mathbb{Z},\ \mathcal{P}_k^\ell = \bigvee_{i=k}^{\ell-1} T^{-i}\mathcal{P},$$

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}_0^n) = \liminf_{n \geqslant 1} \frac{1}{n} H_\mu(\mathcal{P}_0^n),$$

$$h_\mu(T) = \sup_{|\mathcal{P}|<\infty} h_\mu(T, \mathcal{P}).$$

The following characterization of μ_{Haar} in terms of maximal entropy is the main tool we use in the proof of Theorem 2.2, where the map $T : X \to X$ is defined by

$$T(x) = xa(1) = x\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}.$$

THEOREM 2.3 (see [**4**, **5**]). *Let μ be a T-invariant probability measure on X. Then $h_\mu(T) \leqslant h_{\mu_{Haar}}(T) = 1$, and there is an equality if and only if $\mu = \mu_{Haar}$.*

In what follows all partitions of X are implicitly assumed to be finite and measurable. Suppose $\delta_{\Lambda_q}^{\lfloor \ln q \rfloor} \xrightarrow{\text{w}^*} \mu$, $\Lambda_q \subseteq (\mathbb{Z}/q\mathbb{Z})^\times$ for some sequence $q \to \infty$ and let $\mathcal{P}$ be any partition of X such that the boundaries of the atoms of $\mathcal{P}$ have zero μ-measure. This condition implies

$H_\mu(\mathcal{P}_0^m) = \lim_{q\to\infty} H_{\delta_{\Lambda_q}^{\lfloor \ln q \rfloor}}(\mathcal{P}_0^m)$. Our goal in the end is to show that the entropy $h_\mu(T,\mathcal{P})$ is big for a well-chosen partition $\mathcal{P}$, or equivalently that $\frac{1}{m}H_\mu(\mathcal{P}_0^m)$ is big when $m\to\infty$ which is translated to a suitable condition on the entropy of $\delta_{\Lambda_q}^{\lfloor \ln q \rfloor}$.

Recall that for a finite set $\Lambda \subseteq \Gamma\backslash G$, the measure δ_Λ^k is the average of the measures δ_x^k, $x\in\Lambda$, and each of these measures is an average along the T-orbit. Switching the orders of these averages, we get that

$$\delta_\Lambda^k = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\frac{1}{k}\sum_{i=0}^{k-1}\delta_{xa(i)} = \frac{1}{k}\sum_{i=0}^{k-1}T^i\left(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\delta_x\right) = \frac{1}{k}\sum_{i=0}^{k-1}T^i\left(\delta_\Lambda\right).$$

The concavity of the entropy function implies that δ_Λ^k has large entropy if most of the entropies of $T^i(\delta_\Lambda)$ are large, and these are all pushforwards of the same measure δ_Λ. With this idea in mind, we have the following result the proof of which is inspired by the proof of the variational principle in [6].

Lemma 2.4. *Let Y be any measurable space, let $S: Y\to Y$ be some measurable function, $\mathcal{P}$ a partition of Y and μ a probability measure on Y. We denote by $\mu^k = \frac{1}{k}\sum_{i=0}^{k-1}S^i\mu$. Then*

(1) *if $\mu = \sum_1^k a_i\mu_i$ is a convex combination of probability measures μ_i, then $H_\mu(\mathcal{P}) \geqslant \sum_1^k a_i H_{\mu_i}(\mathcal{P})$;*
(2) *for every $n,m\in\mathbb{N}$, we have that*

$$\frac{1}{m}H_{\mu^n}\left(\mathcal{P}_0^m\right) \geqslant \frac{1}{n}H_\mu\left(\mathcal{P}_0^n\right) - \frac{m}{n}\ln|\mathcal{P}|.$$

Proof. (1) Since the function $\alpha: x\mapsto -x\ln(x)$ is concave in $[0,1]$, we obtain that

$$\begin{aligned} H_\mu\left(\mathcal{P}\right) &= \sum_{P\in\mathcal{P}}\alpha\left(\mu\left(P\right)\right) = \sum_{P\in\mathcal{P}}\alpha\left(\sum_1^k a_i\mu_i\left(P\right)\right) \\ &\geqslant \sum_1^k a_i\sum_{P\in\mathcal{P}}\alpha\left(\mu_i\left(P\right)\right) = \sum_1^k a_i H_{\mu_i}\left(\mathcal{P}\right). \end{aligned}$$

(2) Write $n = km + r \leqslant m(k+1)$ where $0\leqslant r < m$. Using subadditivity we get that for $0\leqslant u\leqslant m-1$, we have

$$\begin{aligned} H_\mu\left(\mathcal{P}_0^n\right) &\leqslant H_\mu\left(\mathcal{P}_0^{km+r}\right) \\ &\leqslant \sum_{i=0}^{u-1}H_\mu\left(S^{-i}\mathcal{P}\right) + \sum_{v=0}^{k-1}H_\mu(S^{-(vm+u)}\mathcal{P}_m) + \sum_{i=dm+u}^{dm+m-1}H_\mu(S^{-i}\mathcal{P}) \\ &\leqslant m\log|\mathcal{P}| + \sum_{v=0}^{k-1}H_{S^{vm+u}\mu}(\mathcal{P}_0^m). \end{aligned}$$

Summing over $0\leqslant u\leqslant m-1$, we get that

$$\begin{aligned} mH_\mu\left(\mathcal{P}_0^n\right) - m^2\ln|\mathcal{P}| &\leqslant \sum_{u=0}^{m-1}\sum_{v=0}^{k-1}H_{(S^{vm+u}\mu)}(\mathcal{P}_0^m) \leqslant \sum_{j=0}^{km-1}H_{(S^j\mu)}(\mathcal{P}_0^m) \\ &\leqslant \sum_{j=0}^{n-1}H_{(S^j\mu)}(\mathcal{P}_0^m) \leqslant nH_{\mu^n}\left(\mathcal{P}_0^m\right), \end{aligned}$$

where in the last step we used part (1). It then follows that $\frac{1}{m}H_{\mu^n}(\mathcal{P}_0^m) \geqslant \frac{1}{n}H_\mu(\mathcal{P}_0^n) - \frac{m}{n}\ln|\mathcal{P}|$. □

Corollary 2.5. *Suppose* $\Lambda_q \subset (\mathbb{Z}/q\mathbb{Z})^\times$ *and* $\delta_{\Lambda_q}^{\lfloor \ln q \rfloor} \xrightarrow{\text{w}^*} \mu$ *along some sequence of positive integers* q *for a measure* μ *on* X*. Then, if* $\mathcal{P}$ *is a partition whose atoms have boundary of zero* μ*-measure, then* $h_\mu(T,\mathcal{P}) \geqslant \limsup_{q\to\infty} \frac{1}{\lfloor \ln q \rfloor} H_{(\delta_{\Lambda_q})}(\mathcal{P}_0^{\lfloor \ln q \rfloor})$.

Proof. Follows from Lemma 2.4 and since $\frac{m}{\lfloor \ln q \rfloor}\ln|\mathcal{P}| \to 0$ as $q \to \infty$. □

By the corollary above, we are left with the problem of showing that $\limsup_{q\to\infty} \frac{1}{\lfloor \ln q \rfloor} H_{(\delta_{\Lambda_q})}(\mathcal{P}_0^{\lfloor \ln q \rfloor})$ is big. Suppose that we can show that for every $S \in \mathcal{P}_0^{\lfloor \ln q \rfloor}$, $|S \cap \Lambda_q| \leqslant r$ or in other words $\delta_{\Lambda_q}(S) \leqslant \frac{r}{|\Lambda_q|}$. This would imply

$$\begin{aligned}\frac{1}{\lfloor \ln q \rfloor} H_{\delta_{\Lambda_q}}(\mathcal{P}_0^{\lfloor \ln q \rfloor}) &= \frac{1}{\lfloor \ln q \rfloor} \sum_{S \in \mathcal{P}_0^{\lfloor \ln q \rfloor}} \delta_{\Lambda_q}(S) \ln \frac{1}{\delta_{\Lambda_q}(S)} \\ &\geqslant \frac{1}{\lfloor \ln q \rfloor} \sum_{S \in \mathcal{P}_0^{\lfloor \ln q \rfloor}} \delta_{\Lambda_q}(S) \ln \frac{|\Lambda_q|}{r} = \frac{\ln|\Lambda_q|}{\lfloor \ln q \rfloor} - \frac{\ln r}{\lfloor \ln q \rfloor}. \qquad (2.5)\end{aligned}$$

If $|\Lambda_q|$ is big enough and r is small enough; that is, $\frac{\ln|\Lambda_q|}{\lfloor \ln q \rfloor} - \frac{\ln r}{\lfloor \ln q \rfloor} \to 1$, then we get the lower bound that we wish to establish. We will follow this line of argument with a certain complication that arises. The bound r will basically come from the fact that the diameter of S is small and the points of Λ_q are well separated, but, in fact, one cannot control uniformly the diameter of the atoms of $\mathcal{P}_0^{\lfloor \ln q \rfloor}$. Lemma 2.9 below shows that one can find a partition for which one can do so for most atoms. Before stating Lemma 2.9, we introduce some terminology.

Recall that X is naturally identified with the space of unimodular lattices in the plane. For a lattice $x \in X$, we define the height of x to be

$$\mathrm{ht}(x) = \max\left\{||v||^{-1} : 0 \neq v \in x\right\},$$

and set $X^{\leqslant M} = \{x \in X : \mathrm{ht}(x) \leqslant M\}$ which is compact (similarly, we define $X^{<M}, X^{\geqslant M}, X^{>M}$). Under this notation, $X = \bigcup_1^\infty X^{\leqslant M}$ is σ-compact.

Definition 2.6. For $H \leqslant \mathrm{SL}_2(\mathbb{R})$, define $B_r^H = \{I + W \in H : ||W||_\infty < r\}$. In particular, for $U^+, U^-A \leqslant \mathrm{SL}_2(\mathbb{R})$, we have $B_r^{U^+} = \{I + tE_{1,2} : |t| < r\}$ and $B_r^{U^-A} = \{I + W \in SL_2(\mathbb{R}) : W_{1,2} = 0,\ |W_{i,j}| < r\}$. We also write $B_{\eta,N} = B_{\eta e^{-N}}^{U^+} B_\eta^{U^-A}$, $B_\eta := B_{\eta,0}$.

Definition 2.7. A (finite measurable) partition $\mathcal{P}$ of X is called an (M,η) partition if $\mathcal{P} = \{P_0, P_1, \ldots, P_n\}$ where $P_0 = X^{>M}$ and $P_i \subseteq x_i B_\eta$, $x_i \in X$ for $1 \leqslant i \leqslant n$. If μ is a probability measure on X, then $\mathcal{P}$ is called an (M,η,μ) partition if in addition $\mu(\partial P_i) = 0$ for all i.

Remark 2.8. Given a measure μ, one can construct (M,η,μ)-partitions for arbitrary large M and arbitrary small η in abundance. To see this, we note that $\mu(\partial X^{>M}) = 0$ outside a countable set of numbers M and after defining $P_0 = X^{>M}$ one defines the atoms P_i by a disjointification procedure starting with a finite cover of the compact set $X^{\leqslant M}$ by balls of arbitrarily small radius having μ-null boundary. The point here being is that for a given center x, outside a countable set or radii $\mu(\partial x B_r) = 0$.

Lemma 2.9 is a slight adaptation of Lemma 4.5 from [5]. For convenience, we added the full proof in the Appendix (see also Remark A.3).

Lemma 2.9 (Existence of good partitions [5]). *For any $M > 1$, there exists some $0 < \eta_0(M)$ such that for any $0 < \eta \leqslant \eta_0(M)$ and an $(M, \frac{1}{10}\eta)$ partition $\mathcal{P}$ of X, the following holds: For any $\kappa \in (0,1)$ and any $N > 0$, there exists some $X' \subseteq X^{\leqslant M}$ such that*

(1) *X' is a union of $S_1, \dots, S_l \in \mathcal{P}_0^N$.*
(2) *Each such S_j is contained in a union of at most $C^{\kappa N}$ many balls of the form $zB_{\eta,N}$ with $z \in S_j$ for some absolute constant C.*
(3) *$\mu(X') \geqslant 1 - \mu(X^{>M}) - \mu^N(X^{>M})\kappa^{-1}$ for any probability measure μ on X (where $\mu^N = \frac{1}{N}\sum_{n=0}^{N-1} T_*^n \mu$).*

Lemma 2.9 gives us the tool to produce partitions whose entropies could be controlled in the proof of Theorem 2.2. The last bit of information we need before turning to the proof of Theorem 2.2 is the following separation lemma.

Lemma 2.10 (Good Separation). *Let $p_1, p_2 \in (\mathbb{Z}/q\mathbb{Z})^\times$. If $\Gamma u_{p_1/q}, \Gamma u_{p_2/q} \in zB_{\eta,\lfloor \ln q\rfloor}$, for some $\eta < \frac{1}{100}$, then $p_1 = p_2$.*

Proof. Given the assumption, there exist some $b_1, b_2 \in B_{\eta,\lfloor \ln(q)\rfloor}$ such that $\Gamma u_{p_i/q} = zb_i$, and hence $u_{-p_1/q}\gamma u_{p_2/q} = b_1^{-1}b_2$ for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Applying Lemma A.1, this is contained in $B_{10\eta,\lfloor \ln(q)\rfloor}$. On the other hand, this expression equals to

$$\begin{pmatrix} 1 & -\frac{p_1}{q} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \frac{p_2}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - \frac{p_1}{q}c & b - \frac{p_1}{q}d + \frac{p_2}{q}\left(a - \frac{p_1}{q}c\right) \\ c & d + \frac{p_2}{q}c \end{pmatrix}. \tag{2.6}$$

We conclude that c, the bottom left coordinate, is at most $10\eta < 1$ in absolute value, so that $c = 0$. It then follows similarly that $a = d = 1$. We are then left with the top right coordinate which is $b + \frac{p_2 - p_1}{q}$ which need to be at most $(1+10\eta)10\eta e^{-\lfloor \ln q\rfloor} < \frac{1}{q}$ in absolute value, so we must have that $p_1 = p_2$ and we are done. $\square$

Finally, after collecting all the above information, we are in a position to prove Theorem 2.2 (and by that complete also the proof of Theorem 1.7).

Proof of Theorem 2.2. It is enough to show that μ_{Haar} is the only accumulation point of $\delta_{\Lambda_q}^{\lfloor \ln q\rfloor}$. Let μ be such an accumulation point, which is necessarily T-invariant and by assumption (ii) is a probability measure, and restrict attention to a sequence of positive integers q for which $\delta_{\Lambda_q}^{\lfloor \ln q\rfloor} \xrightarrow{\mathrm{w}^*} \mu$. We shall show that $h_\mu(T) = 1$ and therefore by Theorem 2.3 conclude that $\mu = \mu_{Haar}$ as desired.

By Corollary 2.5, for a partition $\mathcal{P}$ whose atoms have boundary of zero μ-measure, we have that

$$h_\mu(T, \mathcal{P}) \geqslant \limsup_q \frac{1}{\lfloor \ln q\rfloor} H_{\delta_{\Lambda_q}}(\mathcal{P}_0^{\lfloor \ln q\rfloor}). \tag{2.7}$$

Let $\mathcal{P}$ be an (M, η, μ)-partition (see Definition 2.7 and Remark 2.8). Fix $\kappa > 0$ and $N = \lfloor \ln(q)\rfloor$ and let X' be as in Lemma 2.9. If $P \in \mathcal{P}^{\lfloor \ln q\rfloor}$ is such that $P \subseteq X'$, then Lemma 2.9 implies that it can be covered by $C^{\kappa\lfloor \ln q\rfloor}$ sets which by Lemma 2.10 contain at most one element from Λ_q each. This translates to the bound $\delta_{\Lambda_q}(P) \leqslant \frac{1}{|\Lambda_q|}C^{\kappa\lfloor \ln(q)\rfloor}$ and therefore,

$$\begin{aligned}\frac{1}{\lfloor \ln q\rfloor}H_{(\delta_{\Lambda_q})}(\mathcal{P}_0^{\lfloor \ln q\rfloor}) &\geqslant -\frac{1}{\lfloor \ln q\rfloor}\sum_{P\subseteq X'}\delta_{\Lambda_q}(P)\ln(\delta_{\Lambda_q}(P))\\ &\geqslant -\frac{1}{\lfloor \ln q\rfloor}\sum_{P\subseteq X'}\delta_{\Lambda_q}(P)\ln\left(\frac{1}{|\Lambda_q|}C^{\kappa\lfloor \ln q\rfloor}\right)\\ &= \frac{1}{\lfloor \ln q\rfloor}\delta_{\Lambda_q}(X')\left(\ln|\Lambda_q| - \kappa\lfloor \ln q\rfloor \ln(C)\right)\\ &\geqslant \left(1-\delta_{\Lambda_q}^{\lfloor \ln q\rfloor}(X^{\geqslant M})\kappa^{-1}\right)\left(\frac{\ln|\Lambda_q|}{\ln q} - \kappa\ln C\right). \end{aligned} \tag{2.8}$$

Given $\epsilon > 0$, using assumptions (i) and (ii), namely, $\lim \frac{\ln|\Lambda_q|}{\ln q} = 1$ and $\lim_{M\to\infty}\lim_{q\to\infty} \times \delta_{\Lambda_q}^{\lfloor \ln q\rfloor}(X^{\geqslant M}) = 0$, we see that we can choose M to be big enough and κ to be small enough so that for all large enough q the expression on the right in (2.8) is $\geqslant (1-\epsilon)(1-\epsilon)$. We conclude from (2.7) that $h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T,\mathcal{P}) \geqslant 1$ which concludes the proof. □

2.3. *No escape of mass*

Our goal in this section is to prove Theorem 1.5 by showing that the sets $\Lambda_q = (\mathbb{Z}/q\mathbb{Z})^\times$ satisfy the conditions (i) and (ii) of Theorem 1.7. Throughout this section, we set $\Lambda_q = (\mathbb{Z}/q\mathbb{Z})^\times$ and $\mu_q = \delta_{\Lambda_q}$.

We begin with verifying that condition (i) holds which is the content of the following lemma.

Lemma 2.11. *As $q\to\infty$, $\frac{\ln\varphi(q)}{\ln q}\to 1$.*

Proof. Fix q and let $p_i, i = 1,\ldots\omega(q)$ be its prime divisors. Since

$$\varphi(q) = q\prod_{i=1}^{\omega(q)}(1-p_i^{-1}), \tag{2.9}$$

we have that

$$\ln\varphi(q) = \ln q + \sum_{i=1}^{\omega(q)}\ln(1-p^{-1}) \geqslant \ln q + \sum_{i=1}^{\omega(q)}\ln(1/2) = \ln q - \omega(q)\ln 2.$$

We conclude that

$$1 - \frac{\omega(q)}{\ln q}\ln 2 \leqslant \frac{\ln\varphi(q)}{\ln q} \leqslant 1,$$

and since it was shown by Robin in [**12**] that $\omega(q) = O(\frac{\ln q}{\ln\ln q})$, we conclude that $\frac{\ln\varphi(q)}{\ln q}\to 1$ as desired. □

Showing that condition (ii) is satisfied for Λ_q is the content of Lemma 2.14 below. We proceed toward its proof by establishing several lemmas. The following simple lemma basically says that Λ_q is equidistributed on the circle.

Lemma 2.12. *Let q be some integer and $0\leqslant\alpha\leqslant 1$. Then,*

$$|\#\left\{1\leqslant \ell\leqslant \alpha q : \ell\in(\mathbb{Z}/q\mathbb{Z})^\times\right\} - \alpha\varphi(q)| \leqslant 2^{\omega(q)},$$

where $\omega(q)$ is the number of distinct prime factors of q.

Proof. Let p be a prime that divides q and set $U_p = \{1 \leqslant \ell \leqslant \alpha q : p|\ell\}$. We want to find $\lfloor \alpha q \rfloor - |\cup_{p_i} U_{p_i}|$ where p_i are the distinct primes that divide q.

Using inclusion exclusion, we get that

$$\lfloor \alpha q \rfloor - |\cup_p U_p| = \lfloor \alpha q \rfloor - \sum_i |U_{p_i}| + \sum_{i<j} |U_{p_i} \cap U_{p_j}| + \cdots + (-1)^{\omega(q)} |\cap_i U_{p_i}|$$

$$= \lfloor \alpha q \rfloor - \sum_i \left\lfloor \frac{\alpha q}{p_i} \right\rfloor + \sum_{i<j} \left\lfloor \frac{\alpha q}{p_i p_j} \right\rfloor + \cdots + (-1)^{\omega(q)} \left\lfloor \frac{\alpha q}{\prod_i p_i} \right\rfloor .$$

On the other hand, using (2.9), we have that

$$\alpha \varphi(q) = \alpha q \prod_1^{\omega(n)} (1 - \frac{1}{p_i}) = \alpha q - \sum_i \frac{\alpha q}{p_i} + \sum_{i<j} \frac{\alpha q}{p_i p_j} + \cdots + (-1)^{\omega(q)} \frac{\alpha q}{\prod_i p_i}$$

so that

$$|\alpha \varphi(q) - (\lfloor \alpha q \rfloor - |\cup_p U_p|)| \leqslant \sum_{k=0}^{\omega(q)} \binom{\omega(q)}{k} = 2^{\omega(q)}. \qquad \square$$

The following lemma is the heart of the argument yielding the validity of condition (ii) and, in fact, establishes a much stronger non-escape of mass than the one we need, namely it shows that there is no escape of mass for any sequence of measures of the form $a(-t_q)_* \mu_q$ where $q \to \infty$ and t_q is allowed to vary almost without constraint in the interval $[0, \ln q]$; namely it is allowed to vary in $[0, \ln q - 2\omega(q)]$.

Lemma 2.13 (No escape of mass). *Fix some $q \in \mathbb{N}$, $M > 1$ and $0 \leqslant t \leqslant \ln q - 2\omega(q)$. Then*

$$\left|\{p \in (\mathbb{Z}/q\mathbb{Z})^\times : \Gamma u_{p/q} a(t) \in X^{\geqslant M}\}\right| \leqslant \frac{4}{M^2} \varphi(q).$$

Equivalently, $a(-t)_* \mu_q (X^{\geqslant M}) < \frac{4}{M^2}$.

Proof. We say that p is bad if $\Gamma u_{p/q} a(t) \in X_2^{\geqslant M}$. Thus, p is bad if and only if there exists a vector

$$v_p(m,n,t) = (m,n) \begin{pmatrix} 1 & \frac{p}{q} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} = \left(m e^{-t/2}, \left(n + m \frac{p}{q} \right) e^{t/2} \right)$$

such that

$$||v_p(m,n,t)||^2 = m^2 e^{-t} + \left(n + m \frac{p}{q} \right)^2 e^t \leqslant \frac{1}{M^2}, \quad (m,n) \neq (0,0).$$

In particular, this implies $(n + m\frac{p}{q})^2 e^t \leqslant \frac{1}{M^2}$ and $m \leqslant \frac{e^{t/2}}{M}$. We may also assume $m \geqslant 0$ and, in fact, $m \neq 0$, since otherwise $(n + m\frac{p}{q})^2 e^t = n^2 e^t \geqslant n^2 \geqslant 1 > \frac{1}{M^2}$ using the assumption that $t \geqslant 0$. Let us say that p is bad for $m \in [1, \frac{e^{t/2}}{M}]$ if there exists n such that $|n + m\frac{p}{q}| \leqslant \frac{1}{e^{t/2} M}$. We will bound the number of bad integers p by bounding the number of bad p for each $m \in [1, \frac{e^{t/2}}{M}]$.

Given such m and bad p, we can find n such that $|n + m\frac{p}{q}| \leqslant \frac{1}{e^{t/2} M}$ or equivalently $|nq + mp| \leqslant \frac{q}{e^{t/2} M}$. Letting $d_m = \gcd(q,m)$ and writing $q = \tilde{q} d_m,\ m = \tilde{m} d_m$, we get that

$$|\tilde{q} n + \tilde{m} p| \leqslant \frac{\tilde{q}}{e^{t/2} M}. \tag{2.10}$$

We will bound the number of integers p solving (2.10) by considering its meaning in the ring $\mathbb{Z}/\tilde{q}\mathbb{Z}$. Note that $m \leqslant \frac{e^{t/2}}{M} \leqslant \frac{\sqrt{q}}{M}$ so that $\tilde{q} = \frac{q}{(q,m)} \geqslant M\sqrt{q} > 1$. This allows us to consider the group $(\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$ and the natural surjective homomorphism $\pi : (\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$. Furthermore, since $\tilde{m}, p \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$ the meaning of the inequality (2.10) may be interpreted in $(\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$. Namely, if we let $\Omega = \{[a] \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times : |a| \leqslant \frac{\tilde{q}}{e^{t/2}M}\}$, then the bad p for m are exactly $\pi^{-1}(\tilde{m}^{-1}\Omega)$, hence there are at most $|\Omega| \cdot |\ker(\pi)|$ such p. Since π is surjective, we obtain that $|\ker(\pi)| = \frac{\varphi(q)}{\varphi(\tilde{q})}$ and by Lemma 2.12, we get that $|\Omega| \leqslant 2(\frac{1}{e^{t/2}M}\varphi(\tilde{q}) + 2^{\omega(\tilde{q})})$.

We claim that $2^{\omega(\tilde{q})} \leqslant \frac{1}{e^{t/2}M}\varphi(\tilde{q})$. Assuming this claim, the total number of bad integers p (for a fixed m) is at most $|\Omega| \cdot |\ker(\pi)| \leqslant \frac{4}{e^{t/2}M}\varphi(q)$. Since there are $\lfloor \frac{e^{t/2}}{M} \rfloor$ such m, a union bound shows that the number of bad p is at most $\frac{4}{e^{t/2}M}\varphi(q)\frac{e^{t/2}}{M} = \frac{4}{M^2}\varphi(q)$. Thus, to complete the proof, we need only to show $\frac{2^{\omega(\tilde{q})}}{\varphi(\tilde{q})} \leqslant \frac{1}{e^{t/2}M}$. From (2.9), it follows that for any k, $\varphi(k) \geqslant k(\frac{1}{2})^{\omega(k)}$ and so we deduce that

$$\frac{2^{\omega\left(\frac{q}{d_m}\right)}}{\varphi\left(\frac{q}{d_m}\right)} \leqslant \frac{2^{\omega\left(\frac{q}{d_m}\right)}}{(\frac{1}{2})^{\omega\left(\frac{q}{d_m}\right)}\frac{q}{d_m}} = \frac{4^{\omega\left(\frac{q}{d_m}\right)}}{q}d_m \leqslant \frac{e^{2\omega\left(\frac{q}{d_m}\right)}}{q}\frac{e^{t/2}}{M}$$

$$\leqslant \exp\left(t + 2\omega\left(\frac{q}{d_m}\right) - \ln q\right)\frac{1}{e^{t/2}M} \leqslant \frac{1}{e^{t/2}M},$$

where the last inequality follows from the fact that $\omega(q) \geqslant \omega(\frac{q}{d_m})$, and our assumption that $t \leqslant \ln q - 2\omega(q)$ so that $t + 2\omega(\frac{q}{m}) - \ln q \leqslant 0$. □

We now conclude the validity of condition (ii) by averaging the result of Lemma 2.13 over $t \in [0, \ln q]$.

Lemma 2.14. *For any $q > 1$ and any $M > 1$, we have*

$$\mu_q^{\lfloor \ln q \rfloor}\left(X^{<M}\right) \geqslant 1 - \left(\frac{4}{M^2} + O\left(\frac{1}{\ln\ln q}\right)\right).$$

Proof. Using the previous lemma, we get that

$$\mu_q^{\lfloor \ln q \rfloor}\left(X^{\geqslant M}\right) = \frac{1}{\lfloor \ln q \rfloor}\sum_{k=0}^{\lfloor \ln q \rfloor - 1} a(-k)_*\mu_q(X^{\geqslant M})$$

$$\leqslant \frac{1}{\lfloor \ln q \rfloor}\sum_{k=0}^{\lfloor \ln q \rfloor - 2\omega(q) - 1} a(-k)_*\mu_q(X^{\geqslant M}) + \frac{2\omega(q)}{\lfloor \ln q \rfloor} \leqslant \frac{4}{M^2} + \frac{2\omega(q)}{\lfloor \ln q \rfloor}.$$

Finally, it was shown by Robin in [12] that $\omega(q) = O(\frac{\ln q}{\ln\ln q})$, thus completing the proof. □

Proof of Theorem 1.5. By Lemmas 2.11 and 2.14, the two conditions (i) and (ii) of Theorem 1.7 are satisfied for $\Lambda_q = (\mathbb{Z}/q\mathbb{Z})^\times$ yielding the result. □

2.4. *Upgrading the main result*

Theorem 1.5 tells us that the averages $\delta_{\Lambda_q}^{[0,2\ln(q)]}$ where $\Lambda_q = (\mathbb{Z}/q\mathbb{Z})^\times$ converge to the Haar measure. The ergodicity of the Haar measure allows us to automatically upgrade this result to subsets of $(\mathbb{Z}/q\mathbb{Z})^\times$ of positive proportion.

Theorem 2.15. *Let $1 \geqslant \alpha > 0$ and choose $W_q \subseteq (\mathbb{Z}/q\mathbb{Z})^\times$ such that $|W_q| \geqslant \alpha\varphi(q)$ for every q. Then $\delta_{W_q}^{[0,2\ln q]} \xrightarrow{w^*} \mu_{Haar}$.*

Proof. Let μ be an accumulation point of $\delta_{W_{q_i}}^{[0,2\ln(q_i)]}$ for some subsequence q_i (which is necessarily A-invariant). Going down to a subsequence, we may assume $\frac{|W_{q_i}|}{\varphi(q_i)} \to \alpha_0 \geqslant \alpha > 0$ and $\delta_{\Lambda_{q_i} \backslash W_{q_i}}^{[0,2\ln(q_i)]} \to \mu'$ converge. We now have that

$$\mu_{q_i^{[0,2\ln(q)]}} = \frac{|W_{q_i}|}{\varphi(q_i)} \cdot \delta_{W_{q_i}}^{[0,2\ln(q_i)]} + \frac{|\Lambda_{q_i} \backslash W_{q_i}|}{\varphi(q_i)} \cdot \delta_{\Lambda_{q_i} \backslash W_{q_i}}^{[0,2\ln(q_i)]},$$

and taking the limit, we get that

$$\mu_{Haar} = \alpha_0 \mu + (1 - \alpha_0)\mu'.$$

This is a convex combination of A-invariant probability measures with positive α_0. The ergodicity of μ_{Haar} implies that it is extreme point in the set of A-invariant probability measures, hence we conclude that $\mu = \mu_{Haar}$. As this is true for any convergent subsequence of $\delta_{W_q}^{[0,2\ln q]}$, we conclude that it must converge to the Haar measure. $\square$

Once we have the convergence result for any positive proportion sets, we also automatically get a second upgrade and show that almost all choices of sequence $\delta_{p_i/q_i}^{[0,\ln(q_i)]}$ converge.

Proof of Corollary 1.6. Let $\mathcal{F} = \{f_1, f_2, \ldots\}$ be a countable dense family of continuous functions in $C_c(X_2)$. For each $n, q \in \mathbb{N}$, define

$$W_{q,n} = \left\{ p \in (\mathbb{Z}/q\mathbb{Z})^\times : \max_{1 \leqslant i \leqslant n} \left| \left(\delta_{p/q}^{[0,2\ln(q)]} - \mu_{Haar} \right)(f_i) \right| < \frac{1}{n} \right\}.$$

We claim that $\lim_{q\to\infty} \frac{|W_{q,n}|}{\varphi(q)} = 1$ for any fixed n. Otherwise, we can find some $1 \leqslant i \leqslant n$,$\epsilon \in \{\pm 1\}$ and $\alpha > 0$ such that the set

$$V_q = \left\{ p \in (\mathbb{Z}/q\mathbb{Z})^\times : \epsilon \left(\delta_{p/q}^{[0,2\ln(q)]} - \mu_{Haar} \right)(f_i) \geqslant \frac{1}{n} \right\}$$

satisfies $\frac{|V_{q_j}|}{\varphi(q_j)} \geqslant \alpha$ for some subsequence q_j. By Theorem 2.15, we obtain that $\delta_{V_{q_j}}^{[0,2\ln(q_j)]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$, while $\epsilon(\delta_{V_{q_j}}^{[0,2\ln(q_j)]} - \mu_{Haar})(f_i) \geqslant \frac{1}{n}$ for all j, contradiction (note that i, n are fixed).

We conclude that for any n, there exists q_n such that for any $q \geqslant q_n$, $\frac{|W_{q,n}|}{\varphi(q)} \geqslant 1 - 1/n$. Without loss of generality, we may assume that q_n is strictly monotone. We then define for any q, $n_q = \max\{n : q \geqslant q_n\}$. It then follows that $W_q := W_{q,n_q}$ satisfies that $n_q \to \infty$ and $\frac{W_q}{\varphi(q)} \to 1$ as $q \to \infty$. We are left to show $\delta_{p_q/q}^{[0,2\ln(q)]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$ for any choice of sequence $p_q \in W_q$. By the definition of W_q, for any fixed i, we have that $\delta_{p_q/q}^{[0,2\ln(q)]}(f_i) \to \mu_{Haar}(f_i)$, and since $\mathcal{F}$ is dense in $C_c(X_2)$, this claim holds for any $f \in C_c(X_2)$, or in other words, $\delta_{p_q/q}^{[0,2\ln(q)]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$. $\square$

3. *Equidistribution over the adeles*

In this section, we prove Theorem 1.11 which is an enhancement of Theorem 1.5. We establish this equidistribution statement in the adelic space $X_{\mathbb{A}} := \mathrm{PGL}_2(\mathbb{Q}) \backslash \mathrm{PGL}_2(\mathbb{A})$ which we refer to as the adelic extension of $X_{\mathbb{R}} := X_2$.

We shall start in Subsection 3.1 with some general results about locally finite measures and their pushforwards. In particular, we shall prove a ‘compactness’ criterion that roughly states that if the pushforward of a sequence of locally finite measures converges to a probability measure, then it has a subsequence that converges to a probability measure.

In Subsection 3.2, we prove that if μ is an A-invariant lift to $X_{\mathbb{A}}$ of the Haar measure on $X_{\mathbb{R}}$, and satisfies an extra uniformity condition over the finite primes (see Theorem 3.7 for precise definition), then it must be the Haar measure on $X_{\mathbb{A}}$.

Finally, in Subsection 3.3, we show that the limit of translates of the orbit measure $\mu_{\tilde{x}_0 A_{\mathbb{A}}}$ satisfies these condition, thus proving Theorem 1.11.

3.1. *Locally finite measures*

In this section, all the spaces are locally compact second countable Hausdorff spaces. A measure on a space Z is called *locally finite* if every point in Z has a neighborhood with finite measure. Since Z is locally compact, this is equivalent to saying that every compact set has a finite measure. We denote the space of locally finite measures by $\mathcal{M}(Z)$ and the space of homothety classes of such (non-zero) measure by $\mathbb{P}\mathcal{M}(Z)$. Recall that we say that $[\nu_i] \to [\nu]$ for non-zero measures $\nu_i, \nu \in \mathcal{M}(Z)$ if there exist scalars $c_i > 0$ such that $c_i\mu_i \mid_K \xrightarrow{\mathrm{w}^*} \nu \mid_K$ for any compact subset $K \subseteq Z$.

Given two spaces X, Y and a continuous proper map $\pi : X \to Y$, we obtain a map $\mathcal{M}(X) \to \mathcal{M}(Y)$ and its homothethy counterpart $\mathbb{P}\mathcal{M}(X) \to \mathbb{P}\mathcal{M}(Y)$, both of which we shall denote by π_*. We will be interested in lifting convergent sequences from $\mathbb{P}\mathcal{M}(Y)$ to $\mathbb{P}\mathcal{M}(X)$. The next theorem is a type of compactness criterion which assures us that we can lift at least a convergent subsequence. Moreover, if we can show that the limit measure on Y has a unique preimage measure on X, then the convergence in Y will imply a convergence in X.

THEOREM 3.1. *Let $\pi : X \to Y$ be a continuous proper map and let $\nu_i \in \mathcal{M}(X)$ and $\tilde{\nu_i} = \pi_*(\nu_i) \in \mathcal{M}(Y)$. If $[\tilde{\nu_i}] \to [\tilde{\nu}]$ for some probability measure $\tilde{\nu}$ on Y, then $[\nu_{i_k}] \to [\nu]$ for some subsequence i_k and a probability measure ν on X such that $\pi_*(\nu) = \tilde{\nu}$.*

Proof. Multiplying ν_i by suitable scalars, we may assume that $\tilde{\nu}_i \mid_K \xrightarrow{\mathrm{w}^*} \tilde{\nu} \mid_K$ for every compact $K \subseteq Y$. It then follows that $\nu_{i,K} := \nu_i \mid_{\pi^{-1}(K)}$ are finite with uniform bound, since $\nu_{i,K}(X) = \tilde{\nu}_i(K) \to \tilde{\nu}(K) \leqslant 1$. Choose a sequence of compact sets $K_j \nearrow Y$ such that any compact $K \subseteq Y$ is contained in some K_j for some j, which implies the same conditions on $\pi^{-1}(K_j)$. Applying the Banach–Alaoglu theorem, we can find a subsequence i_k such that ν_{i_k,K_j} converges as $k \to \infty$ for every j, which implies $\nu_{i_k} \to \nu$ for some $\nu \in \mathcal{M}(X)$. It then follows that $\pi_*(\nu) = \tilde{\nu}$, and hence ν must be a probability measure. $\square$

3.2. *Lifts of the Haar measure*

For the rest of this section, we fix the following notations. For a set $S \subseteq \mathbb{P}$, where $\mathbb{P}$ is the set of primes in $\mathbb{N}$, we write

$$G_S := \mathrm{PGL}_2(\mathbb{R}) \times \textstyle\prod'_{p\in S} \mathrm{PGL}_2(\mathbb{Q}_p),$$

$$\Gamma_S := \mathrm{PGL}_2(\mathbb{Z}[S^{-1}]),\ \mathbb{Z}\left[S^{-1}\right] := \mathbb{Z}\left[\frac{1}{p} : p \in S\right],$$

where $\prod'_p$ denotes the restricted product with respect to $\mathrm{PGL}_2(\mathbb{Z}_p)$ (which is the standard product if S is finite). Note that Γ_S is embedded as a lattice in G_S via the diagonal map $\gamma \mapsto (\gamma, \gamma, \ldots)$, and we shall denote $X_S := \Gamma_S \backslash G_S$. In case that $S = \mathbb{P}$ or $S = \emptyset$, we will sometimes use the subscript $\mathbb{A}$ (respectively, $\mathbb{R}$) instead, and we remark that $\mathbb{Z}[\mathbb{P}^{-1}] := \mathbb{Q}$ (respectively, $\mathbb{Z}[\emptyset^{-1}] := \mathbb{Z}$). We denote by $\mu_{S,Haar}$ the Haar probability measure on X_S.

We will denote by A_S the full diagonal subgroup in G_S. Note that A is still reserved to the diagonal group with positive entries, namely the matrices $\{\left(\begin{smallmatrix} e^{-t} & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ considered as a subgroup of $\mathrm{PGL}_2(\mathbb{R})$, while $A_{\mathbb{R}} = \{\left(\begin{smallmatrix} \pm e^{-t} & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. For a ring R (usually $\mathbb{R}, \mathbb{Z}, \mathbb{Q}_p$ or $\mathbb{Z}_p$), we will write $U_R = \{u_r = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) : r \in \mathbb{R}\}$ considered as a subgroup in the suitable coordinate of G_S when S contains the corresponding place.

When $S \subseteq S'$, there is a natural projection $X_{S'} \to X_S$ defined as follows. Let

$$H_S := \mathrm{PSL}_2(\mathbb{R}) \times \prod_{p \in S} \mathrm{PGL}_2(\mathbb{Z}_p) \leqslant G_S,$$

$$H'_S := \mathrm{PSL}_2(\mathbb{R}) \times \prod_{p \in S} \mathrm{PSL}_2(\mathbb{Z}_p) \leqslant H_S.$$

Fixing $S \subseteq \mathbb{P}$, it is not hard to show that H_S acts transitively on X_S by using the fact that $\mathbb{Q}_p = \mathbb{Z}_p + \mathbb{Z}[\frac{1}{p}]$, thus leading to the identification $X_S \cong \mathrm{PSL}_2(\mathbb{Z}) \backslash H_S$. This induces the natural projections

$$\pi_S^{S'} : X_{S'} \cong \mathrm{PSL}_2(\mathbb{Z}) \backslash H_{S'} \to \mathrm{PSL}_2(\mathbb{Z}) \backslash H_S \cong X_S \quad \forall S \subseteq S' \subseteq \mathbb{P}.$$

For any S, we have a $\mathrm{PSL}_2(\mathbb{R})$-right action on X_S (and the induced A-action), which commutes with the projections above. Moreover, these projections are easily seen to be proper since the only non-compact part of H_S is $\mathrm{SL}_2(\mathbb{R})$. Thus, we can apply the results from the previous subsection.

The main goal of this section is to lift the equidistribution result on $X_{\mathbb{R}}$ from § 2 to an equidistribution result on $X_{\mathbb{A}}$. We begin by noting that there is a unique G_S-invariant (respectively, H_S) probability measure on X_S, so in order to show that a measure is G_S-invariant, it is enough to show that it is H_S-invariant.

In this subsection, we show that for $S \subseteq \mathbb{P}$ finite, an A-invariant lift of $\mu_{\mathbb{R},Haar}$ to X_S is automatically H'_S-invariant. The main ideas are to first use a maximal entropy argument in order to show that the lift must be $\mathrm{PSL}_2(\mathbb{R})$-invariant, and then show that $\mathrm{PSL}_2(\mathbb{R})$-invariant measures on X_S are actually H'_S-invariant (since we also have $\mathrm{PSL}_2(\mathbb{R})$ 'invariance'). We are then left to show invariance under $H'_S \backslash H_S$, and for that we give the following definition.

Definition 3.2. For $S \subseteq \mathbb{P}$ finite let $\det_S : H_S \to \prod_{p \in S} \mathbb{Z}_p^{\times} / \mathbb{Z}_p^{\times 2}$ be the homomorphism induced from the determinant function $\det : \mathrm{PGL}_2(\mathbb{Z}_p) \to \mathbb{Z}_p^{\times} / \mathbb{Z}_p^{\times 2}$.

As the kernel of this map is exactly H'_S which contains $\mathrm{PSL}_2(\mathbb{Z})$, for any $S \subseteq S'$ (including infinite S'), we use the same notation for the map $\det_S : X_{S'} \to X_S \to H'_S \backslash H_S$.

Remark 3.3. Note that for finite $S \subseteq \mathbb{P}$, the quotient $H'_S \backslash H_S$ is a finite group. Indeed, for each odd prime p, we have that $\mathbb{Z}_p^{\times} / \mathbb{Z}_p^{\times 2} \cong (\mathbb{Z}/p\mathbb{Z})^{\times} / (\mathbb{Z}/p\mathbb{Z})^{\times 2}$ and for $p = 2$, it is isomorphic to $(\mathbb{Z}/8\mathbb{Z})^{\times}$. We conclude, in particular, that H'_S is a unimodular finite index normal subgroup of H_S.

Once we show that these two invariance conditions (the projection to the infinite part $X_{\mathbb{R}}$ and to the finite part $H'_S \backslash H_S$) imply that the lift is the uniform Haar measure on X_S for S finite, we use the structure of restricted products in order to extend this result to infinite S.

Before considering A-invariant measures, we show how to combine right $\mathrm{PSL}_2(\mathbb{R})$-invariance on X_S together with the 'left $\mathrm{PSL}_2(\mathbb{Z})$-invariance' arising from the quotient structure.

Lemma 3.4. *Let $H \leqslant G$ be a unimodular subgroup and assume that either*

(1) *$G = H \times N$, or*
(2) *H is a finite index measurable normal subgroup of G.*

Then any left H-invariant measure is also right H-invariant and vice versa.

Proof. Assume first condition (1) and let μ be a left H-invariant measure on G. Consider the natural product map $C_c(H) \otimes C_c(N) \to C_c(H \times N)$ defined by $(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1)f_2(g_2)$. Using the Stone Weierstrass theorem, we obtain that it has a dense image (in the sup norm); hence, it is enough to show $\mu(R_g(\psi_1 \otimes \psi_2)) = \mu(\psi_1 \otimes \psi_2)$ for any $\psi_1 \in C_c(H)$, $\psi_2 \in C_c(N)$ where R_g (and later on L_g) is the right multiplication by g (respectively left).

If $h \in H$, then $\mu(L_h(\psi_1 \otimes \psi_2)) = \mu((L_h\psi_1) \otimes \psi_2)$ so by the left H-invariance of μ, we learn that $\psi_1 \mapsto \mu(\psi_1 \otimes \psi_2)$ is left H-invariant. The unimodularity of H implies that this map and therefore μ are right H-invariant as well.

Assume now condition (2), namely that $[G : H]$ is finite and we let $\{g_1, \ldots, g_k\}$ be left coset representative of H in G. Let μ be a left H-invariant measure on G. For each i, the measure $\Omega \mapsto \mu_i(\Omega g_i)$ for measurable $\Omega \subseteq H$ is a left H-invariant measure on H; thus, from unimodularity of H, it must also be right H-invariant. Using the normality of H in G and this right invariance, we get that for any $h \in H$ and $\Omega \subseteq H$, we have that $\mu(\Omega g_i) = \mu(\Omega(g_i h g_i^{-1})g_i) = \mu(\Omega g_i h)$. As this is true for all i, it follows that μ is right H-invariant as well. $\square$

LEMMA 3.5. *Let $S \subseteq \mathbb{P}$ be finite and let μ_S be an $\mathrm{PSL}_2(\mathbb{R})$-invariant probability measure on X_S. Then μ_S is H'_S-invariant.*

Proof. Denote by $\mathrm{PSL}_2^{(d)}(\mathbb{Z})$ the diagonal image of $\mathrm{PSL}_2(\mathbb{Z})$ in H_S. The space X_S is a quotient (from the left) by $\mathrm{PSL}_2^{(d)}(\mathbb{Z})$ embedded diagonally, and we consider measures which are invariant (from the right) by $\mathrm{PSL}_2(\mathbb{R})$. We begin with the claim these groups together generate $\overline{< \mathrm{PSL}_2^{(d)}(\mathbb{Z}), \mathrm{PSL}_2(\mathbb{R}) >} = H'_S$. Indeed, it is well known that any element in $\mathrm{SL}_2(\mathbb{Z}_p)$ is generated by $U_{\mathbb{Z}_p}, U^{tr}_{\mathbb{Z}_p}$; hence, this claim follows from the fact that the diagonal embedding of $U_\mathbb{Z} \cong \mathbb{Z}$ is dense in $\prod_{p\in S} \mathbb{Z}_p \cong \prod_{p\in S} U_\mathbb{Z}$ (and similarly for the transpose).

Let $\tilde{\mu}_S$ be the lift of μ_S to H_S, that is, for sets F inside the fundamenal domain, we set $\tilde{\mu}_S(F) = \mu_S(\mathrm{PSL}_2^{(d)}(\mathbb{Z})F)$, and extend this to a left $\mathrm{PSL}_2^{(d)}(\mathbb{Z})$-invariant measure on H_S. Since $H_S = \mathrm{PSL}_2(\mathbb{R}) \times \prod_{p\in S} \mathrm{PGL}_2(\mathbb{Z}_p)$ and $\mathrm{PSL}_2(\mathbb{R})$ is unimodular, we can apply Lemma 3.4 to conclude that $\tilde{\mu}_S$ is left $\mathrm{PSL}_2(\mathbb{R})$ invariant as well, and therefore left $\overline{< \mathrm{PSL}_2^{(d)}(\mathbb{Z}), \mathrm{PSL}_2(\mathbb{R}) >} = H'_S$-invariant. Since H'_S is a finite index normal unimodular subgroup of H_S, applying Lemma 3.4 again, we conclude that $\tilde{\mu}_S$ and therefore μ_S are right H'_S-invariant which completes the proof. $\square$

We can now prove that invariance of the projections of the measure to the finite and infinite places implies the invariance over the adeles. In the following, we consider the actions by $T = \left(\begin{smallmatrix} e^{-1/2} & 0 \\ 0 & e^{1/2} \end{smallmatrix}\right)$ and $U = U_\mathbb{R}$ on the spaces X_S via their images in $\mathrm{PSL}_2(\mathbb{R})$.

THEOREM 3.6 (see [4, Theorems 7.6 and 7.9]). *Fix some finite set $S \subseteq \mathbb{P}$ and let λ be a T-invariant probability measure on X_S. Then $h_\lambda(T) \leqslant 1$ with equality if and only if λ is U-invariant. Similarly, $h_\lambda(T^{-1}) \leqslant 1$ with equality if and only if λ is U^{tr}-invariant (where U^{tr} is the transpose of U).*

THEOREM 3.7. *Let $S \subseteq \mathbb{P}$ be finite and let μ_S be an A-invariant probability measure on X_S, such that $(\pi^S_\mathbb{R})_*\mu_S = \mu_{\mathbb{R},Haar}$. Assume further that the projection $(\det_S)_*\mu_S$ to the finite space $H'_S\backslash H_S$ is the uniform measure. Then $\mu_S = \mu_{S,Haar}$.*

Proof. Since the entropy only decreases in a factor and the Haar measure is U-invariant, an application of Theorem 3.6 shows $1 \geqslant h_{\mu_S}(T) \geqslant h_{\mu_{\mathbb{R},Haar}}(T) = 1$. It follows that $h_{\mu_S}(T) = 1$, and hence μ_S is also U invariant. Repeating the process with T^{-1}, we get that μ_S is $\langle U, U^{tr}\rangle = \mathrm{PSL}_2(\mathbb{R})$-invariant. It now follows from Lemma 3.5 that μ_S is H'_S-invariant.

Let $g_1, \ldots, g_n$ be coset representatives of H'_S in H_S and let $\mu_i(\Omega) := \mu_S(\Omega g_i)$ for $1 \leqslant i \leqslant n$ and $\Omega \subseteq X'_S := \mathrm{PSL}_2(\mathbb{Z}) \backslash H'_S$. All of these measures are H'_S-invariant since μ_S is H'_S-invariant and H'_S is normal in H_S. Letting μ' be the H'_S-invariant probability measure on X'_S, we get that $\mu_i = c_i \mu'$ for some $c_i \geqslant 0$. Any $\Omega \subseteq X_S$ can be written as $\Omega = \bigsqcup \Omega_i g_i$ with $\Omega_i \subseteq X'_S$, and then $\mu_S(\Omega) = \sum c_i \mu'(\Omega_i)$. The projection of μ_S to $H'_S \backslash H_S$ is exactly the probability vector $(c_1, \ldots, c_n)$, hence by assumption $c_i = \frac{1}{n}$ for each i. Given $g \in H_S$, we have that $\Omega g = \bigsqcup \Omega_i g_i g = \bigsqcup \Omega_i h_i g_{j(i)}$ where $h_i \in H'_S$ and $i \mapsto j(i)$ is a permutation. It then follows that $\mu_S(\Omega g) = \sum \frac{1}{n} \mu'(\Omega_i h_i) = \sum \frac{1}{n} \mu'(\Omega_i) = \mu_S(\Omega)$, so we conclude that μ_S is H_S-invariant. $\square$

Finally, we extend this result to the adeles.

THEOREM 3.8. *Let $S \subseteq \mathbb{P}$ be infinite and let μ_S be an A-invariant probability measure on X_S, such that*

(1) *$(\pi^S_{\mathbb{R}})_* \mu_S = \mu_{\mathbb{R},Haar}$, and*
(2) *for any $S' \subseteq S$ finite, the measure $(\det_{S'})_* \mu_S$ is uniform.*

Then $\mu_S = \mu_{S,Haar}$.

Proof. For each finite $S' \subseteq S$, we can pull back the functions in $C_c(X_{S'})$ to $C_c(X_S)$ and the union of these sets over the finite S' spans a dense subset of $C_c(X_S)$. Hence, it is enough to prove that for any such finite set S', $f \in C_c(X_{S'})$ and $g \in H_S$, we have that $\mu_S(g(f \circ \pi^S_{S'})) = \mu_S(f \circ \pi^S_{S'})$. The function $f \circ \pi^S_{S'}$ is already invariant under $g \in G_S$ which are the identity in the $S' \cup \{\infty\}$ places, so it is enough to prove this for $g \in G_{S'}$, and then $g(f \circ \pi^S_{S'}) = g(f) \circ \pi^S_{S'}$. The proof is completed by noting that the measure $\mu_{S'} = (\pi^S_{S'})_* (\mu_S)$ satisfies the conditions of Theorem 3.7, so it is the Haar measure on $X_{S'}$ and hence invariant under $g \in G_{S'}$. $\square$

3.3. *Lifts of orbit measures*

By Theorem 1.5, we know that the averages of the measures $\delta^{[0,2\ln q]}_{p/q}$ converge to the Haar measure on $X_{\mathbb{R}} = X_2 = \mathrm{PGL}_2(\mathbb{Z}) \backslash \mathrm{PGL}_2(\mathbb{R})$ as $q \to \infty$. In this section, we show how to extend these measures to locally finite $A_{\mathbb{R}}$-invariant measures on $X_{\mathbb{R}}$, and relate their averages to projections of single orbit measures in $X_{\mathbb{A}}$.

DEFINITION 3.9. Given a homogeneous space $Z = \Gamma_0 \backslash G_0$, a unimodular group $H < G_0$ and a closed orbit zH, we denote by μ_{zH} the *orbit measure*, namely the pushforward of a restriction of a fixed Haar measure on H to a fundamental domain of $\mathrm{stab}_H(z)$ by the orbit map $h \mapsto zh$. The fact that the orbit is closed and the unimodularity of H imply that the orbit measure is locally finite and H-invariant. Moreover, up to scaling this is the unique H-invariant locally finite measure supported on zH.

For an integer n, we write $\mu_n := \sum_{m \in (\mathbb{Z}/n\mathbb{Z})^\times} \delta_{m/n}$, $\mu_{m/nA} := \mu_{x_0 u_{m/n} A}$ and $\mu_{nA} := \sum_{m \in (\mathbb{Z}/n\mathbb{Z})^\times} \mu_{m/nA}$.

We note that $\frac{1}{2\ln n} \mu_{m/nA} - \delta^{[0,2\ln n]}_{m/n}$ is a positive measure which is supported on the part of the orbit $x_0 u_{m/n} A$ which goes directly to the cusp. Hence, if f is continuous with compact support, we expect that its integral with respect to this difference will be small. This leads us to the following lemma which together with Theorem 1.5 implies Theorem 1.10 as a corollary.

LEMMA 3.10. *For any $f \in C_c(X_{\mathbb{R}})$, we have*

$$\lim_{n \to \infty} \left| \left[\frac{1}{2\ln(n)} \mu_{nA} - \mu^{[0,2\ln n]}_n \right] (f) \right| = 0.$$

Proof. Since f is compactly supported, $supp(f) \subseteq X_2^{\leqslant M}$ for some $M > 0$. For any $m \in (\mathbb{Z}/n\mathbb{Z})^\times$, we have that $\Gamma u_{m/n} a(t) = \Gamma\begin{pmatrix} e^{-t/2} & \frac{m}{n} e^{t/2} \\ 0 & e^{t/2} \end{pmatrix} \in X_2^{>M}$ for all $t \notin [-2\ln(M), 2\ln(n) + 2\ln(M)]$ so that f is zero there, implying

$$\left| \left[\frac{1}{2\ln(n)} \mu_{nA} - \mu_n^{[0,2\ln n]} \right](f) \right| \leqslant \frac{1}{2\ln n} \frac{1}{\varphi(n)} \sum_{(m,n)=1} ||f||_\infty \cdot 4\ln(M)$$

$$= \frac{2\ln(M)}{\ln n} ||f||_\infty \overset{n\to\infty}{\longrightarrow} 0. \qquad \square$$

Proof of Theorem 1.10. The proof that $\mu_{nA} \to \mu_{Haar}$ follows from Lemma 3.10 above and Theorem 1.5. $\square$

We continue to lift these measures to the adeles.

Definition 3.11. We set $G_{\mathbb{A},f} = \prod'_{p\in\mathbb{P}} \mathrm{PGL}_2(\mathbb{Q}_p)$ and consider it as a subgroup of $G_\mathbb{A}$. Similarly, we let $A_{\mathbb{A},f} = A_\mathbb{A} \cap G_{\mathbb{A},f}$.

We now turn to the proof of Theorem 1.11. The strategy will be as follows. Similarly to the real case, if $\tilde{x}_0 = \Gamma_\mathbb{A} \in X_\mathbb{A}$, then $\tilde{x}_0 A_\mathbb{A}$ is a closed orbit and therefore $\mu_{\tilde{x}_0 A_\mathbb{A}}$ is a locally finite A-invariant measure and this remains true if we push this measure by elements from $G_{\mathbb{A},f}$. Thus, if $g_i \in G_{\mathbb{A},f}$ is a sequence satisfying that the projections of $g_i \mu_{\tilde{x}_0 A_\mathbb{A}}$ to $X_\mathbb{R}$ are $\mu_{n_i A}$ with $n_i \to \infty$, then we conclude that the projection of the limit measure to the real place is the uniform Haar measure. The uniformity in the finite places, that is, under the projections $\det_S$ for S-finite will follow from the fact the measure on $A_\mathbb{A}$ is uniform on the finite places.

Since $\mu_{\tilde{x}_0 A_\mathbb{A}}$ is $A_\mathbb{A}$-invariant, a partial limit of $(g_i)_* \mu_{\tilde{x}_0 A_\mathbb{A}}$ will not change if we multiply the g_i by elements of $A_\mathbb{A}$ from the right. Similarly, the limit will be the uniform measure if and only if the limit of $(k_i \cdot g_i)_* \mu_{\tilde{x}_0 A_\mathbb{A}}$ is the uniform measure given a sequence $k_i \in G_\mathbb{A}$ with compact closure. Thus, for a choice of $K = \prod \mathrm{PGL}_2(\mathbb{Z}_p)$, we can consider the g_i in $K \backslash G_{\mathbb{A},f} / A_\mathbb{A}$. The next lemma shows that modulo these groups, the g_i have a very simple presentation.

Definition 3.12. For $m \in (\mathbb{Z}/n\mathbb{Z})^\times$ let $\bar{u}_{m/n} := (u_{m/n}, u_{m/n}, \ldots) \in G_{\mathbb{A},f}$.

Lemma 3.13. *The group $G_{\mathbb{A},f}$ has a decomposition $G_{\mathbb{A},f} = KN'A_{\mathbb{A},f}$ where*

$$K := \prod_{p\in\mathbb{P}} \mathrm{PGL}_2(\mathbb{Z}_p)$$

$$N' := \{\bar{u}_{1/n} : n \in \mathbb{N}\}.$$

Moreover, a sequence $g_i = \bar{u}_{1/n_i}$ in $G_\mathbb{A}/A_\mathbb{A}$ diverges to infinity if and only if $n_i \to \infty$.

Proof. By the Iwasawa decomposition, modulo K from the left and $A_{\mathbb{A},f}$ from the right, any element $g \in G_{\mathbb{A},f}$ can be expressed as $(g_{p_1}, g_{p_2}, \ldots)$ where $g_p = \begin{pmatrix} 1 & \frac{m_p}{p^{l_p}} \\ 0 & 1 \end{pmatrix}, (m_p, p^{l_p}) = 1$, $0 \leqslant m_p < p^{l_p}$ for every p, and $l_p = m_p = 0$ for almost every p. Moreover, by conjugating g_p by a matrix $\begin{pmatrix} k_p & 0 \\ 0 & 1 \end{pmatrix} \in A_{\mathbb{Z}_p} \cap \mathrm{PGL}_2(\mathbb{Z}_p)$, $k_p \in \mathbb{Z}_p^\times$, we can take m_p to be any element in $\mathbb{Z}_p^\times$. Let S be the finite set of primes for which $g_p \notin \mathrm{PGL}_2(\mathbb{Z}_p)$ (that is, $l_p \geqslant 1$) and let $n = \prod p^{l_p} \in \mathbb{N}$. Choosing $m_p = \frac{p^{l_p}}{n}$, we get that $g_p = u_{1/n}$ for all p, thus proving the presentation as $KN'A_{\mathbb{A},f}$.

The second claim follows from the fact that K is compact. $\square$

To prove Theorem 1.11, we are left to show that the limit of $(\pi_{\mathbb{R}}^{\mathbb{A}})_*(\bar{u}_{1/n_i}\mu_{\tilde{x}_0 A_{\mathbb{A}}})$ satisfies the conditions of Theorem 3.8.

Proof of Theorem 1.11. Let $n \in \mathbb{N}$ and consider the measure $\bar{u}_{1/n}\mu_{\tilde{x}_0 A_{\mathbb{A}}}$. Since $stab_{A_{\mathbb{A}}}(\tilde{x}_0) = A_{\mathbb{A}} \cap \mathrm{PGL}_2(\mathbb{Q})$ are the diagonal rational matrices, we obtain that its fundamental domain in $A_{\mathbb{A}}$ is

$$A_{\mathbb{A}}^0 := \left\{ \left(\begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} v_p & 0 \\ 0 & 1 \end{pmatrix}, \dots \right) \in \mathrm{GL}_2(\mathbb{A}) : t \in \mathbb{R},\ v_p \in \mathbb{Z}_p^\times \right\} \leqslant A_{\mathbb{A}}.$$

It follows that $\mu_{\tilde{x}_0 A_{\mathbb{A}}} = \mu_{\tilde{x}_0 A_{\mathbb{A}}^0}$ where the map $a \mapsto \tilde{x}_0 a$ for $a \in A_{\mathbb{A}}^0$ is injective and proper.

Let $N \in \mathbb{N}$ such that $n \mid N$ and define $\psi_N : A_{\mathbb{A}}^0 \to (\mathbb{Z}/N\mathbb{Z})^\times$ by

$$\psi_N : A_{\mathbb{A}}^0 \xrightarrow{\Delta_N} \prod_{p|N} \mathbb{Z}_p^\times \to \prod_{p|N} \left(\mathbb{Z}/p_i^{k_i}\right)^\times \to (\mathbb{Z}/N\mathbb{Z})^\times,$$

where Δ_N is the product over $p \mid N$ of the projections defined by $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mapsto a$.

We claim that for $a \in \psi_N^{-1}(m), m \in (\mathbb{Z}/N\mathbb{Z})^\times$, we have that $\bar{u}_{m'/n} a_f \bar{u}_{-1/n} a_f^{-1} \in \prod U_{\mathbb{Z}_p}$ where a_f is the projection of a to the finite places and m' is the projection of m to $(\mathbb{Z}/n\mathbb{Z})^\times$. Since $(u_{m'/n}, \bar{u}_{m'/n}) \in stab_{A_{\mathbb{A}}}(\tilde{x}_0)$, it follows that

$$\tilde{x}_0 a \bar{u}_{-1/n} = \tilde{x}_0(u_{m'/n} a_\infty, Id, \dots) \cdot \overbrace{\bar{u}_{m'/n} a_f \bar{u}_{-1/n} a_f^{-1}}^{\in \prod_p U_{\mathbb{Z}_p}} a_f,$$

and hence $\pi_{\mathbb{R}}^{\mathbb{A}}(\tilde{x}_0 \psi_N^{-1}(m) \bar{u}_{-1/n}) = x_0 u_{m'/n} A$, namely, up to the projection mod n, the distinct 'cosets' of $\ker \psi_N$ in $A_{\mathbb{A}}^0$ are mapped to the distinct A-orbits in μ_{nA}.

Let $a = (a_\infty, a_{p_1}, a_{p_2}, \dots) \in \psi_N^{-1}(m)$, so that $a_p = \left(\begin{smallmatrix} v_p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $v_p \equiv_{p^{k_p}} m$ where $N = \prod p^{k_p}$. Note that this implies $\frac{m' - v_p}{n} \in \mathbb{Z}_p$ for all p, and then computing the coordinate at the p place, we get

$$u_{m'/n} g_p u_{-1/n} g_p^{-1} = u_{(m'-v_p)/n} \in U_{\mathbb{Z}_p},$$

which proves our claim.

As $\varphi_N^{-1}(m)$ has $\frac{1}{\varphi(N)}$ volume inside the finite places of $A_{\mathbb{A}}^0$, we conclude that

$$(\pi_{\mathbb{R}}^{\mathbb{A}})_*(\bar{u}_{1/n}\mu_{\tilde{x}_0 A_{\mathbb{A}}}) = \frac{1}{\varphi(N)} \sum_{m \in (\mathbb{Z}/N\mathbb{Z})^\times} (\pi_{\mathbb{R}}^{\mathbb{A}})_*(\bar{u}_{1/n}\mu_{\tilde{x}_0 \psi_N^{-1}(m)}) = \frac{1}{\varphi(n)} \sum_{m' \in (\mathbb{Z}/n\mathbb{Z})^\times} \mu_{m'/nA} = \mu_{nA}.$$

In particular, we get that any partial weak limit $\mu_{\mathbb{A}}$ of $\frac{1}{2\ln(n)}(\bar{u}_{1/n})_*\mu_{\tilde{x}_0 A_{\mathbb{A}}}$ is a probability measure such that $\pi_{\mathbb{R}}^{\mathbb{A}}(\mu_{\mathbb{A}}) = \mu_{\mathbb{R},Haar}$.

Fix some finite $S \subseteq \mathbb{P}$ and suppose $8 \cdot \prod_{p \in S} p \mid N$. Since the elements in $U_{\mathbb{Z}_p}$ have determinant 1, with a as above we get that

$$\det_S(\tilde{x}_0 a \bar{u}_{-1/n}) = (v_p/\mathbb{Z}_p^{\times 2} : p \in S) = (m/\mathbb{Z}_p^{\times 2} : p \in S).$$

The map $\Xi(m) = (m/\mathbb{Z}_p^{\times 2} : p \in S)$ for $m \in (\mathbb{Z}/N\mathbb{Z})^\times$ is a well-defined surjective homomorphism, and in particular for each $\sigma \in \prod_{p \in S} \mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}$, we have that

$$\frac{|\Xi^{-1}(\sigma)|}{\varphi(N)} = \frac{1}{|\prod_{p \in S} \mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}|}$$

which depends only on S (and not on N). Letting $\tilde{\mu}_{\sigma,n} := \frac{1}{2\ln(n)} \bar{u}_{1/n}\mu_{\tilde{x}_0(\Xi \circ \psi_N)^{-1}(\sigma)}$, then similar to Lemma 3.10, we can restrict these measure in the infinite place to $0 \leqslant t \leqslant 2\ln(n)$ and obtain probability measure $\hat{\mu}_{\sigma,n}$ which converge together with $\tilde{\mu}_{\sigma,n}$ and to the same measure.

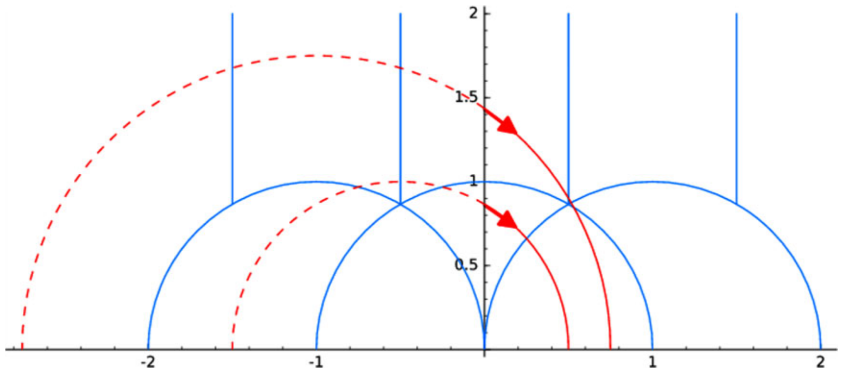

FIGURE 1 (colour online). *The arrows above represent two elements from C_+. C^- is obtained by reflection through the y-axis of C_+.*

Moreover, the limit measures are probability measures since from the one hand, they are limit of probability measures, and, on the other hand, their convex combination is

$$\bar{u}_{1/n}\mu_{\tilde{x}_0 A_{\mathbb{A}}} = \frac{1}{|\prod_{p\in S}\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}|}\sum_\sigma \tilde{\mu}_{\sigma,n},$$

which converge to a probability measure. This implies that the $(\det_S)_*\mu_{\mathbb{A}}$ is the uniform measure which is the second condition needed in Theorem 3.8.

As $S \subseteq \mathbb{P}$ was an arbitrary finite set, Theorem 3.8 implies that $\mu_{\mathbb{A}}$ is the uniform measure on $X_{\mathbb{A}}$. □

4. *From the geodesic flow to the Gauss map*

In this section, we translate the results obtained in § 2 to derive consequences on c.f.e. Using a certain cross section for the flow $a(t)$ on X_2, we relate the partial-orbit measures $\delta_{p/q}^{[0,2\ln(q)]}$ to the normalized counting measures of the finite orbit in [0,1] of p/q under the Gauss map.

We begin by recalling the connection between the c.f.e. and the geodesic flow on the quotient of the hyperbolic plane $\mathbb{H}$ by the action of $\mathrm{PSL}_2(\mathbb{Z})$ by Möbius transformations. We keep the exposition brief and refer the reader to the the book of Einsiedler and Ward [7, Section 9.6] for a detailed account. We bother to repeat many of the things written there as we are mostly concerned with divergent geodesics which form a null set completely ignored in their discussion.

Identifying the unit tangent bundle $T^1\mathbb{H}$ of the hyperbolic plane with $\mathrm{PSL}_2(\mathbb{R})$, we get that every matrix $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$ defines a unique geodesic in $\mathbb{H}$ with endpoints

$$\alpha(g) := \lim_{t\to\infty}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{pmatrix} i = \lim_{t\to\infty}\frac{ae^t i + b}{ce^t i + d} = \frac{a}{c},$$

$$\omega(g) := \lim_{t\to\infty}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2}\end{pmatrix} i = \lim_{t\to\infty}\frac{a\frac{i}{e^t} + b}{c\frac{i}{e^t} + d} = \frac{b}{d}.$$

Following Einsiedler and Ward (see Figure 1), we define

$$C_+ = \{g \in A\cdot \mathrm{SO}_2(\mathbb{R}) : \alpha(g) \leqslant -1 < 0 < \omega(g) < 1\},$$

$$C_- = \{g \in A\cdot \mathrm{SO}_2(\mathbb{R}) : -1 < \omega(g) < 0 < 1 \leqslant \alpha(g)\},$$

$$C = C_+ \cup C_-,$$

considered as subsets of $\mathrm{PSL}_2(\mathbb{R})$.

We leave the following simple proposition to the reader.

PROPOSITION 4.1. *The projection* $\pi : \mathrm{PSL}_2(\mathbb{R}) \to X_2 = \mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R})$ *restricts to a homeomorphism on* C.

Henceforth, we will identify C with $\pi(C)$ and denote points there by $g, \bar{g}$, respectively. This will allow us to speak of the start point $\alpha(\bar{g})$ and end point $\omega(\bar{g})$ for $\bar{g} \in \pi(C)$. For such $\bar{g}$, we will write $sign(\bar{g}) \in \{\pm 1\}$ according to the set C_+ or C_- for which g belongs to.

Our next goal is to show that the Gauss map is a factor of the first return map of the geodesic flow on X_2 to $\pi(C)$. We start by defining a coordinate system on C. Consider the set

$$\tilde{Y} = \left\{(y,z) : y \in (0,1)\,,\ 0 < z \leqslant \frac{1}{1+y}\right\} \times \{\pm 1\} \subseteq \mathbb{R}^2 \times \{\pm 1\}$$

and note that the map from C to $\tilde{Y}$ given by

$$\bar{g} \mapsto \left(|\omega(\bar{g})|, \frac{1}{|\omega(\bar{g}) - \alpha(\bar{g})|}, sign(\bar{g})\right)$$

is a homeomorphism. In what follows we will always use these coordinates.

DEFINITION 4.2. Let $\bar{g} \in \pi(C)$. We define the *return time* $r_C(\bar{g})$ and the *first return map* $T_C(\bar{g})$ to be

$$r_C(\bar{g}) := \min\{t > 0 : \bar{g} \cdot a\,(t) \in \pi(C)\}$$

$$T_C(\bar{g}) := \bar{g} \cdot a\,(r_C(\bar{g})) \in \pi(C).$$

This map is defined only when the forward orbit $\bar{g} \cdot a(t), t > 0$ meets $\pi(C)$. Otherwise, we will write $r_C(\bar{g}) = \infty$.

REMARK 4.3. While it is not trivial, it is not difficult to show that the minimum in the definition of $r_C(\bar{g})$ is well defined (and not just the infinimum). Moreover, $r_C(\bar{g})$ is uniformly bounded from below, that is, $\inf_{g \in \pi(C)} r_C(\bar{g}) > 0$.

We now use the return time map in order to extend our coordinate system.

LEMMA 4.4. *Let* $\hat{Y} = \{(\bar{g}, t) : 0 < t < r_C(\bar{g})\} \subseteq \tilde{Y} \times \mathbb{R}$ *and set* $\theta : (\bar{g}, t) \mapsto \bar{g} \cdot a(t)$. *If* dm *is the restriction of the product measure on* $\tilde{Y} \times \mathbb{R}$ *to* $\hat{Y}$, *then* $\kappa\theta_*(\mathrm{dm}) = \mu_{Haar}$ *for some* $\kappa > 0$, *or equivalently for any* $f \in C_c(X_2)$, *we have that*

$$\int_{X_2} f\,(x)\,\mathrm{d}\mu_{Haar} = \kappa \int_{(y,z,\epsilon)\in Y} \left(\int_{t=0}^{r_C(y,z,\epsilon)} f\,((y,z,\epsilon)\,a\,(t))\,\mathrm{d}t\right) \mathrm{d}\mu_{Leb}. \tag{4.1}$$

Proof. This follows from the proof in [7, Proposition 9.25]. □

The connection between the geodesic flow and the Gauss map is given in the following two lemmas.

LEMMA 4.5 [7, Lemma 9.22]. *Under the identification* $\pi(C) \simeq \tilde{Y}$, *the first return map (where it is defined) is given by*

$$T_C\,(y,z,\epsilon) = (T\,(y)\,, y\,(1 - yz)\,, -\epsilon)\,,$$

where $T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ *is the Gauss map.*

LEMMA 4.6. *Let $0 < x < \frac{1}{2}$ where $x \neq \frac{1}{n}, n \in \mathbb{N}$. The first time that the orbit $\Gamma u_x a(t),\ t \in \mathbb{R}$ meets $\pi(C)$ is at the point $(T(x), x, -1)$ for some $t \geqslant 0$. Similarly, for $\frac{1}{2} < x < 1, x \neq 1 - \frac{1}{n}$, the first meeting is at $(T(1-x), 1-x, 1)$. If $x = \frac{p}{q}$ is rational, then the last time the orbit meets $\pi(C)$ is for some $t \leqslant 2\ln(q)$. Finally, we have that $T^2(x) = T(1-x)$ for $\frac{1}{2} < x < 1$.*

Proof. The proof of the statements involving the first meeting points is essentially the same as the proof of [**7**, Lemma 9.22] and we leave it to the reader. For the statement involving the last meeting time, we note that $\Gamma u_{p/q} a(2\ln(q)) = \Gamma u_{p'/q} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, where $pp' \equiv_q 1$ which as a point in $\mathbb{H}$ is in the standard fundamental domain which points directly up to the cusp; hence, its forward orbit does not pass through $\pi(C)$.

For the second result, let $0 < x < \frac{1}{2}$, so that $x = [0; a_1, a_2, a_3, \ldots]$ with $a_1 \geqslant 2$. We claim that $y = [0; 1, a_1 - 1, a_2, a_3, \ldots]$ is equal to $1 - x$. Indeed, the c.f.e. of y implies

$$y = \frac{1}{1 + \frac{1}{a_1 - 1 + T(x)}} = \frac{1}{1 + \frac{1}{-1 + 1/x}} = 1 - x. \qquad \square$$

The next step is to push measures on X_2 to measures on [0,1] and we do it by lifting functions on $[0,1]$ to functions on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$. The idea is to define the function first on $\pi(C)$ and to thicken it along the A-orbits since $\pi(C)$ has zero measure.

DEFINITION 4.7. Let $r_* = \frac{1}{2} \inf_{g \in \pi(C)} r_C(g) > 0$. For a function $f : [0,1] \to \mathbb{R}$, we define $\tilde{f} : X_2 \to \mathbb{R}$ as follows:

$$\tilde{f}(g) = \begin{cases} \frac{1}{r_*} f(|\omega(g_0)|) & g = g_0 a(t) \ \ s.t.\ g_0 \in \pi(C) \text{ and } 0 < t < r_* \\ 0 & \text{else.} \end{cases}$$

In general, given a probability measure μ on X_2, we would like to define a measure ν on [0,1] by setting $\nu(f) := \mu(\tilde{f})$ for any continuous function f. The problem is that $\mu(\tilde{f})$ is not well defined since $\tilde{f}$ is not continuous with compact support. Fortunately, when $\mu = \delta_x^{[0,R]}$ is any partial orbit measure, $\mu(f)$ is well defined and we obtain the following.

DEFINITION 4.8. For a rational $s = \frac{p}{q} \in \mathbb{Q}$ in the reduced form, we denote by $\mathrm{len}(p/q)$ the first integer i such that $T^i(p/q) = 0$. We define the two measures:

$$\nu_{p/q} = \frac{1}{\mathrm{len}(p/q)} \sum_{i=0}^{\mathrm{len}(p/q)-1} \delta_{T^i(p/q)} \ ; \ \tilde{\nu}_{p/q} = \frac{1}{2\ln(q)} \sum_{i=0}^{\mathrm{len}(p/q)-1} \delta_{T^i(p/q)},$$

LEMMA 4.9. *For any $p \in (\mathbb{Z}/q\mathbb{Z})^\times$ with $q > 2$ and $p \neq 1, q-1$, for any $f : [0,1] \to \mathbb{R}$, we have that $|\delta_{p/q}^{[0,2\ln(q)]}(\tilde{f}) - \tilde{\nu}_{p/q}(f)| < \frac{2}{\ln(q)} ||f||_\infty$*

Proof. Let $t_1 < t_2 < t_3 < \cdots < t_n$ be the times in which the partial orbit $\Gamma u_{p/q} a(t)$, $t \in [0, 2\ln(q)]$ meets $\pi(C)$ and set $\bar{g}_i = (y_i, z_i, \epsilon_i) \in \pi(C)$ to be the corresponding points. It then follows that

$$\left| \delta_{p/q}^{[0,2\ln(q)]}(\tilde{f}) - \frac{1}{2\ln(q)} \sum_1^n f(y_i) \right| \leqslant 2 \frac{||f||_\infty}{2\ln(q)}.$$

By Lemma 4.5, we have that $y_{i+1} = T^i(y_1)$ for all $1 \leqslant i \leqslant n-1$ and by Lemma 4.6, we have that y_1 is either $T(\frac{p}{q})$ when $\frac{p}{q} < \frac{1}{2}$ or $T(1 - \frac{p}{q}) = T^2(\frac{p}{q})$ when $\frac{p}{q} > \frac{1}{2}$, so in any case, the y_i are in the T-orbit of $\frac{p}{q}$. Finally, Lemma 4.6 also tells us that y_n is the last point in the T-orbit of $\frac{p}{q}$, so we conclude that

$$\left|\delta_{p/q}^{[0,2\ln(q)]}(\tilde{f}) - \tilde{\nu}_{p/q}(f)\right| = \left|\delta_{p/q}^{[0,2\ln(q)]}(\tilde{f}) - \frac{1}{2\ln(q)}\sum_{0}^{\mathrm{len}(p/q)-1} f(T^i\left(\frac{p}{q}\right))\right| \leqslant \frac{2}{\ln(q)}||f||_\infty. \qquad \square$$

REMARK 4.10. We note that while $\tilde{\nu}_{p/q}$ appear 'naturally', they are not probability measures. Once we show that such a sequence of measures converge to the probability measure ν_{Gauss}, we immediately get that their probability normalization, namely $\nu_{p/q}$, also converges to ν_{Gauss}.

LEMMA 4.11. *Let* $p_i \in (\mathbb{Z}/q_i\mathbb{Z})^\times$ *such that* $\delta_{p_i/q_i}^{[0,2\ln(q_i)]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$. *Then* $\tilde{\nu}_{p_i/q_i} \xrightarrow{\mathrm{w}^*} 2\ln(2)\kappa\nu_{\mathrm{Gauss}}$ *and therefore* $\frac{\mathrm{len}(p_i/q_i)}{2\ln(q_i)} \to 2\ln(2)\kappa$ *and* $\nu_{p_i/q_i} \xrightarrow{\mathrm{w}^*} \nu_{\mathrm{Gauss}}$.

Proof. Given a segment $I \subseteq [0,1]$ with endpoints $0 \leqslant a < b \leqslant 1$, we have that $\tilde{\chi}_I = \frac{1}{r}\chi_{\Omega_I}$ where

$$\Omega_I = \{\bar{g}_0 a(t) \in X : \bar{g}_0 \in \pi(C)\ ,\ 0 < t < r_*,\ |\omega(g_0)| \in I\}.$$

The boundary of this set is contained in $F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5$, where

$$\begin{aligned}
F_1 &= \pi(C),\\
F_2 &= \pi(C)a(r_*),\\
F_3 &= \{\pi(g)a(t) : g \in A\cdot \mathrm{SO}_2(\mathbb{R}), t \in [0,r_*], |\omega(g)| \in \{0,1\}\},\\
F_4 &= \{\pi(g)a(t) : g \in A\cdot \mathrm{SO}_2(\mathbb{R}), t \in [0,r_*], |\alpha(g)| \in 1\},\\
F_5 &= \{\pi(g)a(t) : g \in A\cdot \mathrm{SO}_2(\mathbb{R}), t \in [0,r_*], |\omega(g)| \in \{a,b\}\}.
\end{aligned}$$

In any case, this is a null set for μ_{Haar}. Since $\delta_{p_i/q_i}^{[0,2\ln(q_i)]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$, for any measurable B with boundary which is μ_{Haar}-null, we have $\delta_{p_i/q_i}^{[0,2\ln(q_i)]}(B) \to \mu_{Haar}(B)$ and, in particular,

$$\delta_{p_i/q_i}^{[0,2\ln(q_i)]}(\Omega_I) \to \mu_{Haar}(\Omega_I) = \kappa\int_{(y,z,\epsilon)\in Y}\int_0^{r_C(y,z,\epsilon)} \chi_{\Omega_I}\mathrm{d}\mu_{Leb} = 2r_*\kappa\int_a^b \frac{1}{1+s}\mathrm{ds}.$$

Applying Lemma 4.9, we obtain that $\tilde{\nu}_{p_i/q_i}(\chi_I) \to 2\ln(2)\kappa\nu_{\mathrm{Gauss}}(\chi_I)$. This result can be extended to any $f \in C[0,1]$ by noting that (1) each such f can be approximated by step function and (2) the measures $\tilde{\nu}_{p/q}$ are uniformly bounded (this follows from the fact that $\mathrm{len}(p/q) \leqslant 2\log_2(q)$).

Now that we have that $\tilde{\nu}_{p_i/q_i} \xrightarrow{\mathrm{w}^*} 2\ln(2)\kappa\nu_{\mathrm{Gauss}}$, evaluating at the constant function 1 produces $\frac{\mathrm{len}(p_i/q_i)}{2\ln(q_i)} \to 2\ln(2)\kappa$ which, in turn, implies $\nu_{p_i/q_i} = \frac{2\ln(q_i)}{\mathrm{len}(p_i/q_i)}\tilde{\nu}_{p_i/q_i} \xrightarrow{\mathrm{w}^*} \nu_{\mathrm{Gauss}}$. $\square$

Proof of Theorem 1.1. By Corollary 1.6, there exist sets $W_q \subseteq (\mathbb{Z}/q\mathbb{Z})^\times$ with $\lim_{q\to\infty}\frac{|W_q|}{\varphi(q)} = 1$, such that for any choice of $p_q \in W_q$, we have that $\delta_{p_q/q}^{[0,2\ln(q)]} \xrightarrow{\mathrm{w}^*} \mu_{Haar}$. Without loss of generality, we may assume $1, q-1 \notin W_q$ (this assumption is not really necessary as this follows automatically since $\delta_{1/q}^{[0,2\ln(q)]}, \delta_{1/q}^{[0,2\ln(q)]}$ cannot converge to μ_{Haar}). The computation $\kappa = \frac{1}{2\zeta(2)}$ will be done in Theorem 4.12 below; hence, applying Lemma 4.11, we obtain that $\frac{\mathrm{len}(p_q/q)}{2\ln(q)} \to \frac{\ln(2)}{\zeta(2)}$ and $\nu_{p_q/q} \xrightarrow{\mathrm{w}^*} \nu_{\mathrm{Gauss}}$ for such sequences. $\square$

Finally, we compute the value of κ. One way of doing it is to note that we already know that $\frac{1}{\varphi(q)}\sum_{p\in(\mathbb{Z}/q\mathbb{Z})^\times}\frac{\mathrm{len}(p/q)}{2\ln(q)}\to 2\ln(2)\kappa$. This limit was computed by Heilbronn in [8] which showed $\kappa=\frac{3}{\pi^2}=\frac{1}{2\zeta(2)}$. A direct computation using the return time map is done in the following theorem.

THEOREM 4.12. *In Equation (4.1), the constant κ is equal to $\frac{3}{\pi^2}=\frac{1}{2\zeta(2)}$.*

Proof. In order to find κ, we compute the return time map and then integrate over $f\equiv 1$. Given the endpoints $\alpha<-1<0<\omega<1$ of g and writing as before $y=\epsilon\omega, z=\epsilon\frac{1}{\omega-\alpha}$, $\epsilon\in\{\pm 1\}$, then $g=\left(\begin{smallmatrix}1-yz & \epsilon y\\ -\epsilon z & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{smallmatrix}\right)$ for some $t\in\mathbb{R}$. In particular, if $g\in C_\pm\subseteq A\cdot\mathrm{SO}_2(\mathbb{R})$, then the rows of g are orthogonal, so that $t=-\ln(\frac{z}{y}(1-yz))/2$. Furthermore, setting $(y',z',\epsilon')=(\frac{1}{y}-\lfloor\frac{1}{y}\rfloor, y(1-yz),-\epsilon)$, we obtain that

$$\begin{pmatrix}-\epsilon\left\lfloor\frac{1}{y}\right\rfloor & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1-yz & \epsilon y\\ -\epsilon z & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & \frac{1}{y}\end{pmatrix}=-\epsilon\begin{pmatrix}1-y'z' & \epsilon'y'\\ -\epsilon'z' & 1\end{pmatrix}.$$

We conclude that $r_C(y,z,\epsilon)=-2\ln(y)-\ln(\frac{z}{y}(1-yz))/2+\ln(\frac{z'}{y'}(1-y'z'))/2$. It then follows that

$$1=2\kappa\int_0^1\int_0^{\frac{1}{1+y}}\left(-2\ln(y)-\ln\left(\frac{z}{y}(1-yz)\right)\Big/2+\ln\left(\frac{z'}{y'}(1-y'z')\right)\Big/2\right)\mathrm{dz}\cdot\mathrm{dy}.$$

Since the map $(y,z)\mapsto(y',z')$ is measure preserving, we conclude that $1=-4\kappa\int_0^1\frac{\ln(y)}{1+y}\mathrm{dy}=4\kappa\frac{\pi^2}{12}$, hence $\kappa=\frac{3}{\pi^2}=\frac{1}{2\zeta(2)}$. □

We finish by giving the proof that for a fixed K, there are very few rationals p/q with $p\in(\mathbb{Z}/q\mathbb{Z})^\times$ such that the coefficients in their c.f.e. are bounded by K.

Proof of Theorem 1.4. Fix some $K>1$ and let

$$\Lambda_{q,K}=\left\{p\in(\mathbb{Z}/q\mathbb{Z})^\times : \text{the entries of the c.f.e. of } \frac{p}{q} \text{ are bounded by K}\right\}.$$

We first claim that there is some $M=M(K)>1$ such that $\delta_{p/q}^{[0,2\ln(q)]}$ is supported in $X_2^{\leqslant M}$ for any $p\in\Lambda_{q,K}$. We give here an elementary proof but the reader may benefit from reviewing [7, Section 9.6] and try to establish this claim by herself. Let $\frac{p}{q}=[0;a_1,a_2,\dots,a_n]$ with $a_i\leqslant K$, and assume $\mathrm{SL}_2(\mathbb{R})u_{p/q}a(t)\in X^{>M}$ for some $0\leqslant t\leqslant 2\ln(q)$. Let $\bar{0}\neq(m,n)\in\mathbb{Z}^2$ such that $||(m,-n)u_{p/q}a(t)||_\infty\leqslant\frac{1}{M}$, or equivalently $|m|\leqslant\frac{e^{t/2}}{M}$ and $|m\frac{p}{q}-n|\leqslant\frac{1}{Me^{t/2}}$. Without loss of generality, we may assume $1\leqslant m\leqslant\frac{e^{t/2}}{M}\leqslant\frac{q}{M}$. Letting $\frac{p_i}{q_i}=[0;a_1,\dots,a_i]$ be the convergents of $\frac{p}{q}$, we have the recursion condition $q_{i+1}=q_ia_{i+1}+q_{i-1}\leqslant(a_{i+1}+1)q_i$. Since $q_n=q$, we obtain that $q_{n-1}\geqslant\frac{q}{a_n+1}\geqslant\frac{q}{K+1}$, so $M>K+1$ implies $m<q_{n-1}$.

Choose k such that $q_{k-1}\leqslant m<q_k\leqslant q_{n-1}\neq q$. Then by the optimality of convergents [7, Proposition 3.3], we get that $|\frac{p}{q}-\frac{p_k}{q_k}|<|\frac{p}{q}-\frac{n}{m}|\leqslant\frac{1}{Mme^{t/2}}$. Furthermore, the convergents satisfy $\frac{1}{2q_{k+1}q_k}<|\frac{p}{q}-\frac{p_k}{q_k}|$ [7, Exercise 3.1.5], and hence

$$\frac{Me^{t/2}}{2}<\frac{q_kq_{k+1}}{m}\leqslant\frac{(a_{k+1}+1)(a_k+1)^2q_{k-1}^2}{m}\leqslant(K+1)^3m\leqslant(K+1)^3\frac{e^{t/2}}{M}.$$

It follows that $M^2<2(K+1)^3$, and therefore the support of $\delta_{p/q}^{[0,2\ln(q)]}$ must be contained in $X^{\leqslant 2(K+1)^2}$.

By the claim that we just proved, the probability measures $\delta_{\Lambda_{q,K}}^{[0,2\ln(q)]}$ are all supported in the compact set $X^{\leqslant 2(K+1)^2}$, so, in particular, they do not exhibit escape of mass. If we also knew that $\frac{\ln|\Lambda_{q,K}|}{\ln(q)} \to 1$, then applying Theorem 1.7, we conclude that $\delta_{\Lambda_{q,K}}^{[0,2\ln(q)]}$ converges to the Haar probability measure, but the limit must also be supported on $X^{\leqslant 2(K+1)^2}$, contradiction. It follows that $\limsup \frac{\ln|\Lambda_{q,K}|}{\ln(q)} < 1$ or equivalently $|\Lambda_{q,K}| = o(q^{1-\varepsilon})$ for some $\varepsilon > 0$. □

Appendix. *The proof of Lemma 2.9*

Before we give the proof, we need some results about hyperbolic balls. Recall from Definition 2.6 that for $H \leqslant SL_2(\mathbb{R})$, we define the H-balls $B_r^H = \{I + W \in H : ||W||_\infty < r\}$. In particular, we have

$$B_r^{U^+} = \{I + \alpha E_{1,2} : |\alpha| < r\}$$

$$B_r^{U^- A} = \{I + W \in SL_2(\mathbb{R}) : W_{1,2} = 0,\ |W_{i,j}| < r\}.$$

We further write $B_{\eta,N} = B_{\eta e^{-N}}^{U^+} B_\eta^{U^- A}$, $B_\eta := B_{\eta,0}$ and $a = \left(\begin{smallmatrix} e^{-1/2} & 0 \\ 0 & e^{1/2} \end{smallmatrix}\right)$ (so that $aB_r^{U^+}a^{-1} = B_{r/e}^{U^+}$).

Lemma A.1. *Let $H \leqslant G$ be any subgroup. We have the following.*

1. $(B_K^H)^{-1} = B_K^H$.
2. $B_{K_1}^H B_{K_2}^H \subseteq B_{2(K_1+K_2)}^H$ *whenever* $K_1, K_2 < 1$.
3. *Suppose* $r^+, r^- < \frac{1}{4}$. *Then* $B_{r^-}^{U^- A} B_{r^+}^{U^+} \subseteq B_{2r^+}^{U^+} B_{2r^-}^{U^- A}$.
4. *Suppose* $r^+, r^- < \frac{1}{4}$. *Then* $gB_{r^+}^{U^+} g^{-1} \in B_{2r^+}^{U^+} B_{6r^-}^{U^- A}$ *for every* $g \in B_{r^-}^{U^- A}$.
5. *Suppose* $r^+, r^- < \frac{1}{16}$ *and* $x, y \in \Gamma\backslash G$. *Then*

$$y \in xB_{r^+}^{U^+} B_{r^-}^{U^- A} \Rightarrow xB_{r^+}^{U^+} B_{r^-}^{U^- A} \subseteq yB_{8r^+}^{U^+} B_{6r^-}^{U^- A}.$$

Proof. (1) Follows from the fact that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ for matrices of determinant 1.

(2) Follows from the identity $(I + W_1)(I + W_2) = I + (W_1 + W_2) + W_1W_2$ and the fact that $||W_1W_2||_\infty \leqslant 2||W_1||_\infty||W_2||_\infty$.

(3) Suppose $|u|, |v|, |w| < r^-$ and $|x| < r^+$. Then

$$\begin{pmatrix} 1+u & 0 \\ v & 1+w \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{x(1+u)}{1+w+vx} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+u-\frac{x(1+u)}{1+w+vx}v & 0 \\ v & 1+w+vx \end{pmatrix},$$

which is in $B_{2r^+}^{U^+} B_{2r^-}^{U^- A}$.

(4) Using the previous parts, we get that

$$gB_{r^+}^{U^+} g^{-1} \subseteq B_{r^-}^{U^- A} B_{r^+}^{U^+} B_{r^-}^{U^- A} \subseteq B_{2r^+}^{U^+} B_{2r^-}^{U^- A} B_{r^-}^{U^- A} \subseteq B_{2r^+}^{U^+} B_{6r^-}^{U^- A}.$$

(5) Using the previous parts, $y = xh^+h^-$ with $h^+ \in B_{r^+}^{U^+}$ and $h^- \in B_{r^-}^{U^- A}$, we have that

$$xB_{r^+}^{U^+} B_{r^-}^{U^- A} = y\left(h^-\right)^{-1}\left(h^+\right)^{-1} B_{r^+}^{U^+} B_{r^-}^{U^- A} \subseteq yB_{r^-}^{U^- A} B_{4r^+}^{U^+} B_{r^-}^{U^- A}$$

$$\subseteq yB_{8r^+}^{U^+} B_{2r^-}^{U^- A} B_{r^-}^{U^- A} \subseteq yB_{8r^+}^{U^+} B_{6r^-}^{U^- A}.$$

□

Lemma A.2. *There is some constant C such that for all $0 < r_1, r_2$ small enough, $x \in X_2$ and $Y \subseteq xB_{r_1}^{U^+} B_{r_2}^{U^- A}$, there are $y_1, \ldots, y_C \in Y$ such that $Y \subseteq \bigcup y_i B_{r_1/e}^{U^+} B_{r_2/e}^{U^- A}$.*

Proof. We first prove a similar claim in $\mathrm{SL}_2(\mathbb{R})$, that there exists a constant C_1 such that for all $0 < r_1, r_2$ small enough and $R \geqslant 1$, we can find $x_1, \dots, x_{C_1 R^3} \in \mathrm{SL}_2(\mathbb{R})$ such that $B_{r_1}^{U^+} B_{r_2}^{U^- A} \subseteq \bigcup x_i B_{r_1/R}^{U^+} B_{r_2/R}^{U^- A}$. Since $U^+ \cong \mathbb{R}$, given $R' \geqslant 1$, we can find $O(R')$ elements $g_i \in B_{r_1}^{U^+}$ such that $B_{r_1}^{U^+} \subseteq \bigcup g_i B_{r_1/R'}^{U^+}$, and similarly, we can find $O((R')^2)$ elements $h_j \in B_{2r_2}^{U^- A}$ such that $B_{r_2}^{U^-} \subseteq \bigcup h_j B_{r_2/R'}^{U^- A}$. Applying Lemma A.1, we obtain that

$$B_{r_1}^{U^+} B_{r_2}^{U^-} \subseteq \bigcup_{i,j} g_i B_{r_1/R'}^{U^+} h_j B_{r_2/R'}^{U^- A} = \bigcup_{i,j} g_i h_j \left(h_j^{-1} B_{r_1/R'}^{U^+} h_j\right) B_{r_2/R'}^{U^- A}$$

$$\subseteq \bigcup_{i,j} g_i h_j B_{2r_1/R'}^{U^+} B_{6r_2/R'}^{U^- A} B_{r_2/R'}^{U^- A} \subseteq \bigcup_{i,j} g_i h_j B_{2r_1/R'}^{U^+} B_{14r_2/R'}^{U^- A}.$$

Choosing $R' = 14R$ finishes the claim.

We now transfer this result to X_2. Let $r_1, r_2 > 0$ small enough, $x \in X_2$ and $Y \subseteq x B_{r_1}^{U^+} B_{r_2}^{U^- A}$. Setting $R = 8e$, we can find $O(R^3) = O(1)$ many $x_i \in \mathrm{SL}_2(\mathbb{R})$ such that $x B_{r_1}^{U^+} B_{r_2}^{U^-} \subseteq \bigcup x \cdot x_i B_{r_1/R}^{U^+} B_{r_2/R}^{U^- A}$. Choose y_i such that $y_i \in Y \cap x \cdot x_i B_{r_1/R}^{U^+} B_{r_2/R}^{U^- A}$ if this set is not empty and otherwise choose some $y_i \in Y$ arbitrarily. Since $y_i \in x \cdot x_i B_{r_1/R}^{U^+} B_{r_2/R}^{U^- A}$, applying Lemma A.1 (5), we get that

$$x \cdot x_i B_{r_1/R}^{U^+} B_{r_2/R}^{U^- A} \subseteq y_i B_{8r_1/R}^{U^+} B_{6r_2/R}^{U^- A} = y_i B_{r_1/e}^{U^+} B_{r_2/e}^{U^- A},$$

which completes the proof. $\square$

Proof of Lemma 2.9. Choose $\eta_0(M) > 0$ to be small enough so that Lemmas A.1 and A.2 will be applicable and that the map $g \mapsto xg$ from $B_\eta \to \Gamma \backslash G$ is injective for all $x \in X^{\leqslant M}$. Let $\mathcal{P} = \{P_0, \dots, P_n\}$ be an (M, η) partition.

Consider the function $f(x) = \frac{1}{N} \sum_0^{N-1} 1_{X^{>M}}(T^i x)$ and note that this function is constant on each $P \in \mathcal{P}_N$.

Setting $X' = X^{\leqslant M} \cap \{x : f(x) \leqslant \kappa\}$, we obtain that

$$1 \leqslant \mu\left(X^{>M}\right) + \mu\left(\{f(x) > \kappa\}\right) + \mu\left(X'\right) \leqslant \mu\left(X^{>M}\right) + \kappa^{-1} \int f(x)\, \mathrm{d}\mu + \mu\left(X'\right)$$

$$= \mu\left(X^{>M}\right) + \kappa^{-1} \mu^N\left(X^{>M}\right) + \mu\left(X'\right),$$

thus proving part (3) in the theorem.

For $S \in \mathcal{P}_N$, $S \subseteq X'$ set $V_m = \left|\left\{0 \leqslant i \leqslant m \;\middle|\; T^i(S) \subseteq X^{>M}\right\}\right|$. Let C be the constant from Lemma A.2. We claim that $S \subseteq \bigcup_1^{C^{|V_m|}} y_i B_{\eta,N}$ with $y_i \in S$ for any $0 \leqslant m \leqslant N$, and the lemma will follow by setting $m = N-1$. For $m = 0$, let $y \in S \subseteq P_i \subseteq x_i B_{\frac{\eta}{10}}$ for some $i \geqslant 1$, so by Lemma A.1, $S \subseteq y B_\eta$, thus proving the case for $m = 0$.

Assume $S \subseteq \bigcup_1^{C^{|V_m|}} y_i B_{\eta,m}$ with $y_i \in S$ for $m < N-1$ and we prove for $m+1$.

- Suppose first $T^{m+1} S \subseteq X^{\leqslant M}$ so that $T^{m+1} S \subseteq P_j \subseteq x_j B_{\frac{\eta}{10}}$ for some $j \geqslant 1$. This case will be complete if $S \cap y_i B_{\eta,m} = S \cap y_i B_{\eta,m+1}$ for every i. Indeed, Lemma A.1 implies $T^{m+1} S \subseteq x_j B_{\frac{\eta}{10}} \subseteq y_i a^{(m+1)} B_\eta$, so if $y_i g \in S$ with $g \in B_{\eta,m}$, then

$$\left[y_i a^{(m+1)}\right] a^{-(m+1)} g a^{(m+1)} = y_i g a^{(m+1)} \in T^{m+1} S' \subseteq y_i a^{(m+1)} B_\eta.$$

By the assumption on the injectivity radius, we conclude that $g \in B_{\eta,m} \cap a^{(m+1)} B_\eta a^{-(m+1)} = B_{\eta,m+1}$, which is what we wanted to show.

- Suppose now $T^{n+1} S \subseteq X^{>M}$. By Lemma A.2, for each i, we have that $S \cap y_i B_{\eta,m} \subseteq \bigcup_{j=1}^C \tilde{y}_i^{(j)} B_{\frac{\eta}{e},m} \subseteq \bigcup_{j=1}^C \tilde{y}_i^{(j)} B_{\frac{\eta}{e},m+1}$ with $\tilde{y}_i^j \in S$, which completes this case and the proof. $\square$

Remark A.3. In the original proof of Lemma 4.5 from [**5**], there was a slight inaccuracy in the final argument where the center of the balls $yB_{\eta,m}$ was not shown to be inside S. This inaccuracy is resolved in Lemma A.2.

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