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Dirichlet Shapes of Unit Lattices and Escape of Mass

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We study the collection of points on the modular surface obtained from the logarithm embeddings of the groups of units in totally real cubic number fields (which we term Dirichlet shapes of unit lattices). We conjecture that this set is dense and show that its closure contains countably many explicit curves and give a strategy to show that it has non-empty interior. The results are obtained by constructing explicit families of orders (generalizing the so called “simplest cubic fields") and calculating their groups of units. We also address the question of escape of mass for the compact orbits of the diagonal group associated to these orders.

1 Introduction

This article originates from an attempt to understand concrete examples of sequences of compact orbits for the diagonal group $A < \mathrm{SL}_3(\mathbb{R})$ on the space of lattices $X \overset{\mathrm{def}}{=} \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$. We investigate two seemingly unrelated questions one can ask about such orbits. The first deals with a certain number theoretic invariant of a compact A-orbit (the shape of the unit lattice attached to it), and the second deals with the distribution of the orbit in X. While we do not see any obvious connection between the two questions, this article is concerned with constructing families of lattices in X for

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which both of these questions can be answered. Nevertheless, see the last paragraph of section 1.2 and Remark 2.12 for a heuristics regarding why we expect that when one exhibits a construction for the first question the second question is likely to be answered as well.

1.1 Dirichlet shapes of unit lattices

We begin by explaining the first question which we find most interesting and seems to deserve further study. It is well known (see section 2) that given an order $\mathcal{O}$ in a totally real cubic number field, one can construct out of it a three-dimensional lattice with a compact A-orbit whose geometric shape is governed by the shape of the group of units $\mathcal{O}^\times$.

More precisely, let $\{\sigma_i\}_1^3$ denote the embeddings of the field into the reals. Dirichlet's unit theorem states that if we denote for $\omega \in \mathcal{O}^\times$, $\psi(\omega) \stackrel{\text{def}}{=} (\log|\sigma_1(\omega)|, \log|\sigma_2(\omega)|, \log|\sigma_3(\omega)|)$, then ψ maps $\mathcal{O}^\times$ to a two-dimensional lattice in the plane $\mathbb{R}^3_0 \stackrel{\text{def}}{=} \left\{\mathbf{t} \in \mathbb{R}^3 : \sum_1^3 t_i = 0\right\}$. We define the *Dirichlet shape* $\Delta_{\mathcal{O}^\times}$ of the unit lattice $\mathcal{O}^\times$ to be the corresponding point on the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. This correspondence is defined as follows: one chooses a similarity map to identify $\mathbb{R}^3_0$ and $\mathbb{R}^2$ which maps $\psi(\mathcal{O}^\times)$ to a unimodular lattice in $\mathbb{R}^2$, that is, to a point in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Since the similarity is only well-defined up to rotation, we obtain a well-defined point in $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We set

$$\Omega \stackrel{\text{def}}{=} \{\Delta_{\mathcal{O}^\times} \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} : \mathcal{O} \text{ is an order in a totally real cubic number field}\}.$$

We wish to propose the following conjectures.

Conjecture 1.1. (1) The closure $\overline{\Omega}$ in the modular surface is non-compact.
(2) The closure $\overline{\Omega}$ in the modular surface has non-empty interior.
(3) The set Ω is dense in the modular surface. □

Despite the fact that the above conjectures are natural, as far as we know there is virtually nothing in the literature about them. In personal communication with Andre Reznikov we learned that questions which are similar in spirit to the above were also suggested by Margulis and Gromov and that numerical experiments seem to support Conjecture 1.1. We provide modest progress toward Conjecture 1.1((2)) and prove that $\overline{\Omega}$ contains countably many explicit curves illustrated in Figure 1. For more details see Theorem 1.4. In Figure 2, we plotted the Dirichlet shapes of the unit lattices of rings of integers of totally real fields with bounded discriminants.

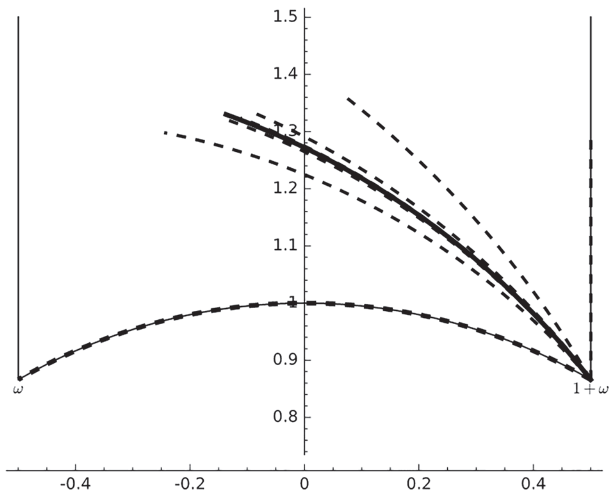

Fig. 1. The dashed curves are cluster points of Dirichlet shapes of rings of the form $\mathbb{Z}[\theta]$, where θ is a unit, which we construct. The thick curve in the middle is (one of the countably many) limits of the dashed curves.

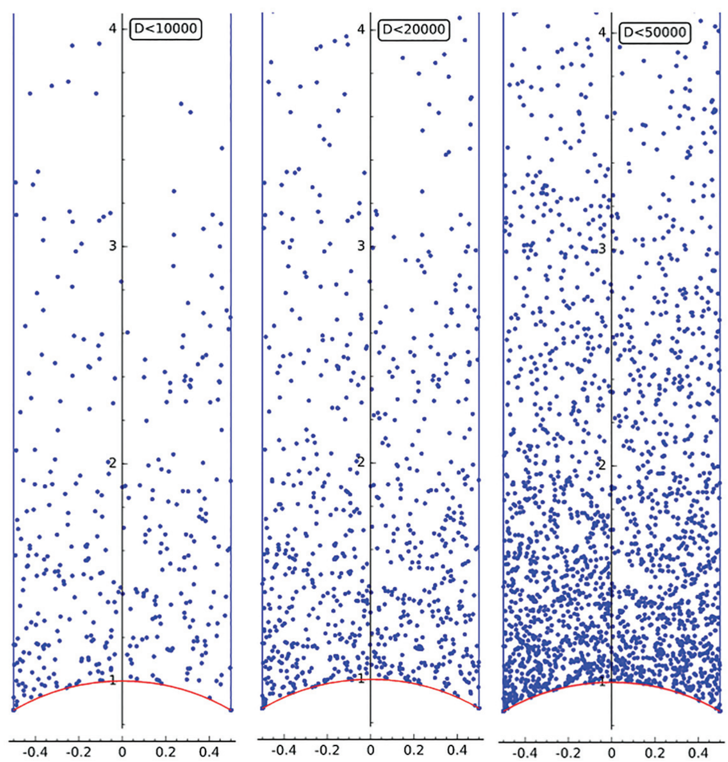

Fig. 2. Above are sage plots of Dirichlet shapes of the unit lattices of rings of integers of totally real fields with discriminant bounded from above by 10,000, 20,000, and 50,000, respectively.

In Problem 1.5 we give a strategy of how to reduce Conjecture 1.1((2)) to a certain problem in finding enough solutions to some congruence conditions.

1.2 Escape of mass

We now describe the second question we study. One of the major open questions in homogeneous dynamics is to understand the space of A-invariant and ergodic probability measures on X. Conjecturally, this space is composed of periodic measures only. Here, a probability measure is called *periodic* if it is L-invariant and supported on a single orbit $Lx \subset X$, where $L < \mathrm{SL}_3(\mathbb{R})$ is a closed subgroup. In such a case we denote this measure by μ_{Lx} and say that the orbit Lx is a *periodic orbit*. In fact, in the above example, due to scarcity of closed groups $A < L < \mathrm{SL}_3(\mathbb{R})$, the A-invariant and ergodic periodic measures in this space are μ_X — the unique $\mathrm{SL}_3(\mathbb{R})$-invariant probability measure on X — and the ones corresponding to periodic A-orbits (which is a synonym for compact A-orbits). Apart from describing what are the A-invariant and ergodic probability measures on X, it is desirable to understand the topology of this space. In particular, what can be said about the weak* accumulation points of sequences μ_{Ax_n} of periodic A-invariant measures supported on compact A-orbits. The question that we study for a sequence μ_{Ax_n} is that of *partial* or *full escape of mass*. We say that a sequence μ_{Ax_n} exhibits *c-escape of mass* for $0 < c \leq 1$ if any weak* accumulation point μ of μ_{Ax_n} satisfies $\mu(X) \leq 1 - c$. We say that it exhibits *full escape of mass* if it exhibits 1-escape of mass, that is, if μ_{Ax_n} converges to the zero measure.

The reason for the number theoretic interest in periodic A-orbits is that they correspond to full modules in totally real cubic number fields as will be discussed later. Our aim in this direction is to review and construct particular examples of such full modules and establish partial or full escape of mass of the corresponding orbits. In practice, what we do is exhibit a family of cubic polynomials $\{f_i(x) : i \in I\} \subset \mathbb{Z}[x]$ which depend on some parameter $i \in I$ (such that $f_i(x)$ is irreducible and totally real) and discuss the periodic A-orbit corresponding to the order $\mathbb{Z}[\theta_i]$, where θ_i is a root of $f_i(x)$, as the parameter varies. One might expect that if the polynomials are chosen carefully, then conclusions regarding the orbits could be derived.

The fact that these orders contain 1 implies in turn that the corresponding A-orbit contains a point “close to infinity" (because we normalize by the discriminant). When this is coupled with the fact that these orbits are in some sense “small" we obtain the desired escape of mass. See Remark 2.12 for a heuristic reason regarding why one can explicitly construct only “small" orbits.

1.3 Structure of the paper and results

In section 2 we give the general notation and correspondence between full modules of general orders in number fields and periodic A-orbits in X. In particular, we will give a condition on the relation between the discriminant and the unit group that will be sufficient to produce escape of mass.

As stated before, we are interested in lattices arising from rings of the form $\mathbb{Z}[\theta]$, where θ is the root of some monic irreducible polynomial $f(x)$. In particular, we will be interested in the case where the units of $\mathbb{Z}[\theta]$ are generated by elements of the form $a\theta - b, c\theta - d$ (and -1).

We start this investigation in section 3 and show in Lemma 3.1 that a necessary condition for $a\theta - b, c\theta - d$ to be units in $\mathbb{Z}[\theta]$ is that $a^3 f(\frac{b}{a}) = \pm 1$ and similarly $c^3 f(\frac{d}{c}) = \pm 1$, which is a solution to two integral equations in the coefficients of f.

In sections 3.2 and 3.3 we will show how to construct monic cubic polynomials $f_{a,b,c,d,t}(x)$ (all parameters being integers) which satisfy these conditions. Moreover, we will show that there are infinitely many such polynomials (parameterized by t) whenever a, b, c, d satisfy a simple congruence conditions, and we will give some examples for such a, b, c, d.

In section 3.4, we fix the parameters a, b, c, d and take $|t| \to \infty$. In this case, for $|t|$ big enough, the polynomials $f_{a,b,c,d,t}(x)$ will be irreducible and not only will the elements $a\theta - b, c\theta - d$ be integral units, they will actually form a system of fundamental units (i.e., they generate the unit group together with -1). More precisely, we have the following result which is a direct consequence of Theorems 3.12 and 3.13.

Theorem 1.2. Fix $a, b, c, d \in \mathbb{Z}$ such that $a, c \neq 0$, $\frac{b}{a} \neq \frac{d}{c}$, $a \neq \pm c$, and there exists a monic cubic polynomial $h(x) \in \mathbb{Z}[x]$ satisfying $a^3 h\left(\frac{b}{a}\right) = \epsilon_1$, $c^3 h\left(\frac{d}{c}\right) = \epsilon_2$, where $\epsilon_i = \pm 1$. We denote $h_t(x) = h(x) + tg(x)$, $g(x) = (ax - b)(cx - d)$, where $t \in \mathbb{Z}$. Then the following holds

(1) For all $|t|$ big enough, the polynomial $h_t(x)$ is totally real and irreducible.
(2) Setting θ_t to be a root for h_t, for all $|t|$ big enough the unit group of $\mathbb{Z}[\theta_t]$ is generated by $\{a\theta_t - b, c\theta_t - d, -1\}$.
(3) As $|t| \to \infty$ the Dirichlet shape of the unit lattice converges to the regular triangles lattice.
(4) As $|t| \to \infty$, the orbits corresponding to the orders $\mathbb{Z}[\theta_t]$ exhibit full escape of mass. □

We remark two things: (1) What stands behind Theorem 1.2 is that the family of polynomials h_t is controlled by a degree 2 polynomial g, and as $|t|$ increases, we can approximate the roots of h_t using the roots of g. (2) As mentioned above, the existence of a polynomial h used to jumpstart Theorem 1.2 is guaranteed by a simple congruence condition on the parameters a, b, c, d.

The phenomena described in Theorem 1.2 are similar in nature to what happens in [10] and in particular, the shape of the regular triangle lattice is the only possible limit shape. It turns out that in order to create new limit shapes one needs to vary the parameters a, b, c, d with t. This approach is implemented in section 3.5. In fact, to simplify matters we concentrate on the case where $c = 1, d = 0$ (i.e., that θ is an integral unit), and take $a := a_t, b := b_t$ to increase to infinity as $|t| \to \infty$. As mentioned earlier, in order to construct the relevant polynomial for such a_t, b_t they need to satisfy a simple congruence condition which we now define.

Definition 1.3. We say that a pair of integers (a, b) is a *mutually cubic root pair* if $a^3 \equiv_b 1$ and $b^3 \equiv_a 1$, that is, $a \mid b^3 - 1$ and $b \mid a^3 - 1$. A sequence (a_t, b_t) is called a *mutually cubic root sequence* if (a_t, b_t) is a mutually cubic root pair for any $t \in \mathbb{N}$ or $\mathbb{Z}$. □

Given a mutually cubic root sequence, we are able to construct a family of orbits which exhibit (at least) partial escape of mass. Furthermore, we will also compute the Dirichlet shapes of the unit lattices and their limit as $|t| \to \infty$. Unlike the case with a, b, c, d fixed, here the limit Dirichlet shapes will not necessarily be the regular triangles lattice.

Theorem 1.4. Let (a_t, b_t) be a mutually cubic root sequence and suppose that the limits $\tilde{a} = \lim_{t\to\infty} \frac{\log|a_t|}{\log|t|}$ and $\tilde{b} = \lim_{t\to\infty} \frac{\log|b_t|}{\log|t|}$ exist and satisfy $0 \le \tilde{a} \le \tilde{b}$. Then $\overline{\Omega} \subseteq \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ contains the image of the curve

$$\gamma(r) = \frac{1 + 2r\tilde{a} + \left(1 + r\tilde{b} + 2r\tilde{a}\right)\omega}{1 + r\tilde{a} + \left(r\tilde{a} - r\tilde{b}\right)\omega} \quad r \in \left[0, \min(\frac{1}{3\tilde{a}}, \frac{1}{\tilde{b}})\right],$$

where $\omega = e^{\frac{2\pi i}{3}}$. □

We note that the ratios $\frac{\tilde{a}}{\tilde{b}} = \lim_{t\to\infty} \frac{\log|a_t|}{\log|b_t|}$ (thought of as points in $P^1(\mathbb{R})$) and the curves are in 1-1 correspondence. In section 3.6, we will show how to produce infinitely many examples of mutually cubic root sequences (a_t, b_t), which in turn produce countably many distinct limits of the form $\lim_{t\to\infty} \frac{\log|a_t|}{\log|b_t|}$. The Dirichlet shape of unit lattices produced by these orders can be seen in Figure 1. As a consequence to the previous theorem, it is straightforward to see that a positive solution to the following problem will imply Conjecture 1.1((2)).

Problem 1.5. Let $\Lambda \subset \mathbb{P}(\mathbb{R})$ be the set of all the possible ratios $\frac{\tilde{a}}{\tilde{b}}$, where $\tilde{a} = \lim \frac{\log a_t}{\log t}$, $\tilde{b} = \lim \frac{\log b_t}{\log t}$ (not both zero), and (a_t, b_t) is a mutually cubic root sequence. Is the interior of Λ non-empty? □

We remark that in Corollary 3.27 we show that Λ has infinitely many accumulation points.

1.4 Comparison with earlier results

This work is a succession of the discussion in [10] in which the second named author addressed the above questions in regards to certain sequences of compact A-orbits (in any dimension). In that discussion, all sequences of compact A-orbits exhibited full escape of mass but more interestingly, the Dirichlet shapes of unit lattices there converged to a fixed shape which in dimension 3 is the shape of the regular triangle lattice which corresponds to the point $\omega = \exp(\frac{2\pi i}{3})$ on the corner of the fundamental domain in Figure 1. Interestingly, in the present work, when we produce examples of orders for which the Dirichlet shapes of unit lattices converge to a shape not equal to ω we only manage to establish partial escape of mass.

The problem of finding generators for the group of units $\mathcal{O}^\times$ is classical. Explicit examples of computations may be found in [4, 7, 8, 12], and most notably in the spirit of the current discussion, in [2] where it is shown that the curve at the bottom of the fundamental domain in Figure 1, $e^{2\pi i\theta}, \theta \in [\pi/3, 2\pi/3]$ is contained in $\overline{\Omega}$.

2 Preliminaries

We now set up the number theoretic notation and terminology needed for our discussion. For the general background from number theory, see for example [9].

Let $\mathbb{K}/\mathbb{Q}$ be a totally real number field of degree n. A *full module* M in $\mathbb{K}$ is an abelian subgroup $M = sp_{\mathbb{Z}}\{\alpha_1, \ldots, \alpha_n\} \le \mathbb{K}$ such that $\mathbb{Q}M = \mathbb{K}$. An *order* in $\mathbb{K}$ is a full module which is also a unital ring. We denote by $\mathcal{O}_{\mathbb{K}}$ the ring of integers which is the unique maximal order in $\mathbb{K}$. Let $\sigma_1, \ldots, \sigma_n : \mathbb{K} \to \mathbb{R}$ be the n distinct real embeddings of $\mathbb{K}$. The homomorphism $\varphi : \mathbb{K} \to \mathbb{R}^n$ defined by $\varphi(\alpha) := (\sigma_i(\alpha))_1^n$ is an embedding which sends any full-module $M < \mathbb{K}$ to a lattice in $\mathbb{R}^n$. The *discriminant* D_M of M is defined as the square of the covolume of $\varphi(M)$. We denote by $D_{\mathbb{K}}$ the discriminant of of the ring of integers $\mathcal{O}_{\mathbb{K}}$. Given a full module M, we define the *associated order* of M to be $\mathcal{O}_M := \{\alpha \in \mathbb{K} \mid \alpha M \subseteq M\}$ and denote by $\mathcal{O}_M^\times$ the group of units of $\mathcal{O}_M$. Note that M is itself an order if and only if $M = \mathcal{O}_M$. Let $\psi : \mathbb{K} \to \mathbb{R}^n$ be defined by $\psi(\alpha) = (\log|\sigma_i(\alpha)|)_1^n$. Since the norm of a unit is ± 1, $\psi(\mathcal{O}_M^\times) \subset \mathbb{R}_0^n := \left\{x \in \mathbb{R}^n : \sum_1^n x_i = 0\right\}$. Dirichlet's unit theorem says that $\psi(\mathcal{O}_M^\times)$ is a lattice in $\mathbb{R}_0^n$. A collection $\left\{\alpha_j\right\}_1^{n-1} \subset \mathcal{O}_M^\times$ is a *system of fundamental units* if $\left\{\psi(\alpha_j)\right\}_1^{n-1}$ forms a basis for $\psi(\mathcal{O}_M^\times)$. The Regulator R_M of M is defined as the covolume of the projection of $\psi(\mathcal{O}_M^\times)$ into any copy of $\mathbb{R}^{n-1}$ spanned by the axis in $\mathbb{R}^n$. Equivalently, if $\left\{\alpha_j\right\}_1^{n-1} \leq \mathcal{O}_M^\times$ is a system of fundamental units then R_M is the determinant of any $(n-1) \times (n-1)$ submatrix of of the matrix $\left(\log\left|\sigma_i\left(\alpha_j\right)\right|\right)$, where $1 \leq i \leq n$ and $1 \leq j \leq n-1$. If $\left\{\alpha_j\right\}_1^{n-1}$ is just a set of independent units, we shall call this determinant the *relative regulator*.

We now restrict our attention to orders in totally real cubic fields. The following theorem and its corollary will give us the tool to prove that a pair of units is a system of fundamental units. This was used also in [2].

Theorem 2.1 (Cusick [1]). For an order in a totally real cubic number field of discriminant D and regulator R, one has $\frac{R}{\log^2\left(\frac{D}{4}\right)} \geq \frac{1}{16}$. In particular, for any sequence of such orders with discriminants and regulators D_i, R_i respectively, we have $\liminf_{i\to\infty} \frac{R_i}{\log^2(D_i)} \geq \frac{1}{16}$. □

We remark that the formulation of this result in [1] is for maximal orders but that the proof works verbatim for a general order.

Corollary 2.2. Let $\mathbb{K}$ be a totally real cubic field, and let $M \leq \mathbb{K}$ be an order with discriminant D and regulator R. If $\{\alpha_1, \alpha_2\} \leq M^\times$ is an independent set of units with relative regulator R' such that $\frac{R'}{\log^2\left(\frac{D}{4}\right)} < \frac{1}{8}$, then they must be a fundamental set. □

Proof. Follows immediately from the previous theorem and the fact that $R'/R = [\mathcal{O}_{\mathbb{K}} : \langle\alpha_1, \alpha_2, -1\rangle]$. ■

We briefly describe the relation between full modules and compact A-orbits in the space of lattices. The space of unimodular lattices is identified as usual with $X := \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ and we denote by $A \leq \mathrm{SL}_n(\mathbb{R})$ the subgroup of positive diagonal matrices.

Given a full module M in a totally real degree n number field $\mathbb{K}$ with embeddings $\{\sigma_i\}_1^n$, we denote $L_M := D_M^{-\frac{1}{2n}}\varphi(M) \in X$. The compactness of the orbit AL_M is a consequence of Dirichlet's theorem as we now explain. This compactness is equivalent to the statement that $\mathrm{stab}_{\mathbb{R}_0^n}(L_M) := \left\{x \in \mathbb{R}_0^n : \exp(x)L_M = L_M\right\}$ is a lattice in $\mathbb{R}_0^n$, where here $\exp : \mathbb{R}_0^n \to A$ is given by $\exp(x) := \mathrm{diag}\,(e^{x_1}, \ldots, e^{x_n})$. It is straightforward that for $\alpha \in \mathbb{K}^\times$, $L_{\alpha M} = a_\alpha L_M$ where $a_\alpha := \mathrm{diag}\,(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$ on the diagonal. Therefore, if $\alpha \in \mathcal{O}_M^\times$ then $a_\alpha L_M = L_M$ and $\det a_\alpha = \pm 1$ (because α has norm ± 1). If all the values of $\sigma_i(\alpha)$ are positive then $\psi(\alpha) \in \mathrm{stab}_{\mathbb{R}_0^n}(L_M)$. In fact, the converse is also true (see [3, 5, 6, 11]),that is, if we set $\mathcal{O}_M^{\times,+} := \left\{\alpha \in \mathcal{O}_M^\times : \forall i, \sigma_i(\alpha) > 0\right\}$, then

$$\psi(\mathcal{O}_M^{\times,+}) = \mathrm{stab}_{\mathbb{R}_0^n}(L_M).$$

Now since $\psi(\mathcal{O}_M^{\times,+})$ is a finite index subgroup of $\psi(\mathcal{O}_M^\times)$, and the latter is a lattice in $\mathbb{R}_0^n$ by Dirichlet's theorem, we conclude that $\psi(\mathcal{O}_M^{\times,+})$ is a lattice as well.

Remark 2.3. We note two things. First, it is a classical fact (that we will not use), that all compact A-orbits are of the form AL_M for some full module M as above (see any of [3, 5, 6, 11]). Second, although when studying the orbit AL_M, the lattice $\psi(\mathcal{O}_M^{\times,+})$ is a more natural object of study, it is much more natural from the number theoretic point of view to work with the lattice $\psi(\mathcal{O}_M^\times)$. In Corollary 2.6 we will show that for the purpose of escape of mass, passing from $\psi(\mathcal{O}_M^{\times,+})$ to $\psi(\mathcal{O}_M^\times)$ is harmless. □

We turn now to present the necessary tools to establish the escape of mass in our results. For a more thorough discussion, the reader is referred to [10].

Definition 2.4. Let $L \in X$ be a unimodular lattice.

(1) We define the height of L to be

$$ht(L) = \left(\min\{||v|| \mid 0 \neq v \in L\}\right)^{-1} = \max\left\{||v||^{-1} \mid 0 \neq v \in L\right\}.$$

(2) For $H \geq 0$ we define $X^{\leq H}$ (respectively, $<, \geq, >$) by

$$X^{\leq H} = \{L \in X \mid ht(L) \leq H\}.$$ □

The sets $X^{\leq H}$ are compact and $X = \bigcup_H X^{\leq H}$. The statement that a sequence of periodic A-orbits Ax_k exhibits c-escape of mass for $0 < c \leq 1$ is equivalent to the statement that for any $H > 0, \epsilon > 0$ and any k large enough $\mu_{Ax_k}(X^{\geq H}) \geq c - \epsilon$.

The minor difference between $\psi(\mathcal{O}_M^{\times,+})$ and $\psi(\mathcal{O}_M^{\times})$ does not play any role in the discussion of escape of mass because of the following.

Lemma 2.5. Let M be a full module in a totally real number field as above. The height map $h : \mathbb{R}_0^n/\psi(\mathcal{O}_M^{\times,+}) \to \mathbb{R}$ given by $h(x) := ht(\exp(x)L_M)$ factors through $\mathbb{R}_0^n/\psi(\mathcal{O}_M^{\times})$. □

Proof. if $x, y \in \mathbb{R}_0^n$ are such that $x - y \in \psi(\mathcal{O}_M^{\times})$ then there exists $\alpha \in \mathcal{O}_M^{\times}$ and a diagonal ± 1 matrix J_α such that $\exp(x-y) = J_\alpha a_\alpha$. Since J_α acts as an isometry on $\mathbb{R}^n$ and $a_\alpha L_M = L_M$ we get that

$$h(x) = ht(\exp(x)L_M) = ht(\exp(y)\exp(x-y)L_M)$$
$$= ht(\exp(y)J_\alpha a_\alpha L_M) = ht(J_\alpha \exp(y)L_M) = ht(\exp(y)L_M) = h(y).$$ ■

Corollary 2.6. Let AL_M be a compact A-orbit as above, let F be a fundamental domain for $\psi(\mathcal{O}_M^{\times})$ in $\mathbb{R}_0^n$, let λ denote the Lebesgue measure on $\mathbb{R}_0^n$ and let μ_{AL_M} be the periodic A-invariant probability measure on the orbit AL_M. Let $h : \mathbb{R}_0^n \to \mathbb{R}$ be the height function $h(x) := ht(\exp(x)L_M)$. Then, for any $H > 0$ we have $\mu_{AL_M}(X^{>H}) = \frac{1}{\lambda(F)} \lambda(\{x \in F : h(x) > H\})$. □

In practice, the way we prove escape of mass is by using the above corollary: We find a good fundamental domain for the unit lattice $\psi(\mathcal{O}_M^{\times})$ on most of which we have control on the height.

Henceforth we restrict our discussion to dimension $n = 3$. We now explain how to we choose good fundamental domains for $\psi(\mathcal{O}_M^{\times})$ in which we control the height in a good enough manner. We need to introduce a few definitions first.

Definition 2.7. A set $\Phi = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbb{R}_0^3$ is called a *simplex set* if $span_{\mathbb{R}}\Phi = \mathbb{R}_0^3$ and $\sum_1^3 \alpha_i = 0$.

Denote by $\Delta_\Phi = span_{\mathbb{Z}}\{\Phi\}$ the lattice generated by Φ and by W_Φ the set

$$W_\Phi = \left\{ \sum_1^3 \lambda_i \alpha_i \;\middle|\; \{\lambda_1, \lambda_2, \lambda_3\} = \left\{0, \frac{1}{3}, \frac{2}{3}\right\} \right\}.$$ □

Since any simplex set Φ is a linear image of the simplex set giving rise to the regular triangle lattice, we conclude from Figure 3 that conv (W_Φ) is a fundamental domain for the lattice Δ_Φ.

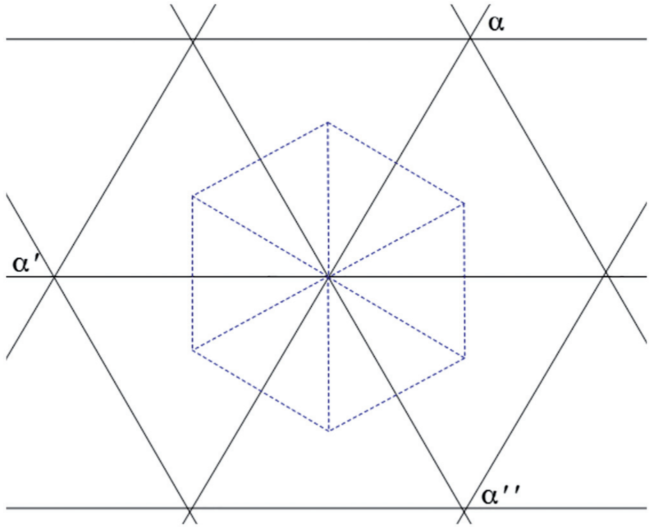

Fig. 3. The inner hexagon with dashed lines, which equals $\mathrm{conv}(W_\Phi)$, is a fundamental domain for the regular triangles lattice generated by $\Phi = \{\alpha, \alpha', \alpha''\}$. The six points W_Φ are its vertices.

Definition 2.8. For a vector $v = \left(v^{(1)}, v^{(2)}, v^{(3)}\right) \in \mathbb{R}^3_0$ we write $\lceil v \rceil = \max v^{(i)}$. For a set $\tilde{\Phi}$ we denote $\left\lceil \tilde{\Phi} \right\rceil = \max_{v \in \tilde{\Phi}} \lceil v \rceil$. □

Lemma 2.9. Let Φ_i be a sequence of simplex sets such that $\Phi_i \subset \psi(\mathcal{O}_i^\times)$, where $\mathcal{O}_i$ is a sequence of distinct orders in totally real cubic fields. Then, $\lceil W_{\Phi_i} \rceil \to \infty$. □

Proof. It is not hard to show that if the covolume of the lattice Δ_Φ goes to ∞ then so does $\lceil W_\Phi \rceil$. The lemma now follows because the covolume of Δ_{Φ_i} is proportional to the regulator $R_{\mathcal{O}_i}$ and it is well known that there are only finitely many orders with regulators under a given bound, the lemma follows. ■

In Theorem 2.11 below we see that the term $\lceil W_\Phi \rceil$ controls the escape of mass. Before stating this theorem we need the following.

Definition 2.10. Let M be a full module in a totally real cubic number field. We say that a simplex set $\Phi \subseteq \psi(\mathcal{O}_M^\times)$ is (R, r)-*tight* for $R \geq 1$ and $0 \leq r \leq 1$ if $\exp(r \lceil W_\Phi \rceil) \leq ht(L_M) R$. □

Theorem 2.11. Let $R, r > 0$ be fixed. Let M_i be a sequence of full modules in totally real cubic number fields with distinct associated orders $\mathcal{O}_i$. Let $\Phi_i \subseteq \psi(\mathcal{O}_i^\times)$ be simplex sets which are (R, r)-tight. Then the sequence of periodic A-orbits AL_{M_i} exhibits r^2-escape of mass. □

Proof. Consider the height function $h : \mathrm{conv}(W_{\Phi_i}) \to \mathbb{R}$ given by $h(x) = ht(\exp(x)L_{M_i})$. We show that for any $H > 0$, any $r_0 < r$, and any large enough i, $r_0 \cdot \mathrm{conv}(W_{\Phi_i}) \subset \left\{x \in \mathrm{conv}(W_{\Phi_i}) : h(x) > H\right\}$ and so, by Corollary 2.6, since $r_0^2 = \frac{\lambda(r_0 \cdot \mathrm{conv}(W_{\Phi_i}))}{\lambda(\mathrm{conv}(W_{\Phi_i}))}$, the sequence exhibits r^2-escape of mass as claimed.

To this end, fix $H > 0, r_0 < r$, and let $x \in r_0 \cdot \mathrm{conv}\left(W_{\Phi_i}\right) \subseteq \mathbb{R}_0^3$ for some $0 \leq r_0 < r$ and write $x = \sum_{\beta \in W_{\Phi_i}} \lambda_\beta \beta$ as a convex combination. For $0 \neq v \in L_{M_i}$ of norm $ht(L_{M_i})^{-1}$ we get that

$$|| \exp(x)\, v|| \leq ||v|| \max_{1 \leq \ell \leq 3} (\exp(x_\ell)) \leq ||v|| \max_{\ell} (\exp(\sum_{\beta \in W_{\Phi_i}} \lambda_\beta \beta^{(\ell)}))^{r_0}$$

$$\leq ||v|| \left(\exp \left\lceil W_{\Phi_i} \right\rceil\right)^{r_0} \leq R \left(\exp \left\lceil W_{\Phi_i} \right\rceil\right)^{r_0 - r}.$$

It follows that for $x \in r_0 \cdot \mathrm{conv}(W_{\Phi_i})$, $h(x) > R^{-1}(\lceil W_{\Phi_i} \rceil)^{r - r_0}$ and the latter expression is greater than H for i large enough since $\left\lceil W_{\Phi_i} \right\rceil \to \infty$ by Lemma 2.9. ■

Remark 2.12. As explained to us by Elon Lindenstrauss we remark here that the three-dimensional lattices that we build here have very small A-orbits in the following sense: The volume of the orbit is (up to a constant) the regulator and by the class number formula the biggest the regulator can get is roughly square root of the discriminant. On the other hand, if one can express explicitly a fundamental set of units of an order $\mathbb{Z}[\theta]$, where θ has f as minimal polynomial, then it is very likely (and indeed happens in our analysis), that the units in this fundamental set would be polynomial expressions in the roots of f and thus the regulator would be a logarithmic expression in the roots of f. In turn, the discriminant is polynomial in the roots of f and therefore the regulator is logarithmic in the discriminant. This might serve as a heuristics for why in any explicit construction of sequences of orders in which we are able to exhibit a fundamental set of units we also observe some escape of mass. □

3 Construction of Cubic Orders

3.1 Generalizing the simplest cubic fields

To motivate the constructions presented below we begin by reviewing a classical family of cubic fields. These are known as the *simplest cubic fields*. They get their name from the ease in the computation of their integer ring, integral units, and other important algebraic invariants. This example was used by Cusick in [1] to show that the limit $\lim_{i\to\infty}\frac{R_i}{\log^2(D_i)} = \frac{1}{16}$ in Theorem 2.1 can be attained. This family consists of the fields $\mathbb{K}_t = \mathbb{Q}(\theta_t)$ for $0 \le t \in \mathbb{Z}$, $t \not\equiv_9 3$, where θ_t is a root of the polynomial

$$f_t(x) = x^3 - tx^2 - (t+3)x - 1 = \left(x^3 - 3x - 1\right) - t \cdot x\,(x+1)\,.$$

It is well known that for infinitely many t we have that $\mathcal{O}_{\mathbb{K}_t} = \mathbb{Z}\,[\theta_t]$, and the unit group is generated by $-1, \theta_t, \theta_t + 1$ (see e.g., [13]). The fact that these are indeed units is easy to see from the polynomials f_t. The norm of θ_t is just the free coefficient of f_t, namely $f_t(0) = -1$, so that θ_t is an integral unit. The norm of $\theta_t + 1$ is the free coefficient of $f_t(x-1)$, namely it is $f_t(-1) = (-1)^3 + 3 - 1 = 1$ so it is again a unit. Note that the norms are independent of t, since $0, -1$ are roots of $x(x+1)$.

With this idea, we construct below polynomials $f_{a,b,c,d,t}(x)$ giving rise to orders of the form $\mathbb{Z}\,[\theta]$ such that their unit group is generated by $a\theta - b,\ c\theta - d$, and -1 (for t large enough). These types of orders were studied in [4, 8, 12] with some restrictions on a, b, c, d and in greater generality in [7], though with a rather complex set of conditions on a, b, c, d. We will give a simple congruence condition on a, b, c, d that will ensure that the group of units is indeed generated by the above, and furthermore, in section 3.6 we will show how to construct infinitely many tuples (a, b, c, d) which satisfy our congruence conditions.

Given $a, b, c, d \in \mathbb{Z}$ which satisfy some mild conditions, we classify the family of polynomials $f(x) \in \mathbb{Z}[x]$ having a root θ such that $a\theta - b, c\theta - d$ are units in the ring $\mathbb{Z}[\theta]$.

Lemma 3.1. Let $f(x) \in \mathbb{Z}\,[x]$ be a monic, cubic irreducible polynomial with root θ. Then for $a, b \in \mathbb{Z},\ a \neq 0$ we have that $N(a\theta - b) = -a^3 f(\frac{b}{a})$. In particular, $a\theta - b$ is a unit in $\mathbb{Z}\,[\theta]$ if and only if $a^3 f(\frac{b}{a}) = \pm 1$. Additionally, if this is the case we must have that $\gcd(a, b) = 1$. □

Proof. We recall that $a\theta - b$ is a unit if and only if $N(a\theta - b) = \pm 1$ and this norm is minus the free coefficient of the monic minimal polynomial of $a\theta - b$ which is $a^3 f\left(\frac{x+b}{a}\right)$. It follows that $a\theta - b$ is a unit if and only if $\pm 1 = -N(a\theta - b) = a^3 f\left(\frac{b}{a}\right)$.

A necessary condition is that $(a, b) = 1$. Indeed, if $a = a'd$ and $b = b'd$ with $d > 1$, then

$$N(a\theta - b) = N(d \cdot \left(a'\theta - b'\right)) = d^3 N(a'\theta - b') \neq \pm 1,$$

and therefore $a\theta - b$ is not a unit. ■

Remark 3.2. Clearly, we have that $N(a\theta - b) = -N(-a\theta + b)$ in cubic extensions, and furthermore multiplying by -1 elements from a system of fundamental units will produce a system of fundamental units. Thus, when searching for units we will always assume that the norm is -1, and in the notation of the previous lemma we have that $a^3 f(\frac{b}{a}) = 1$. □

Lemma 3.3. Let $a, b, c, d \in \mathbb{Z}$ be given and assume that $a, c \neq 0$, $ad - bc \neq 0$ and that $\gcd(a, b) = \gcd(c, d) = 1$. Let

$$\mathcal{F} = \mathcal{F}_{a,b,c,d} = \left\{ h \in \mathbb{Z}[x] : \begin{array}{l} h \text{ is a monic cubic irreducible polynomial with root} \\ \theta \text{ such that both } a\theta - b \text{ and } c\theta - d \text{ are units} \\ \text{in } \mathbb{Z}[\theta] \text{ of norm -1} \end{array} \right\}.$$

Then $h \in \mathcal{F}$ implies that $\mathcal{F} \subseteq \{h_t(x) = h(x) + t\,(ax - b)\,(cx - d) : t \in \mathbb{Z}\}$. □

Proof. If $f \in \mathcal{F}$, then $f(x) - h(x)$ is a degree 2 polynomial (since both are monic). Also, from Lemma 3.1 we conclude that $\frac{b}{a}, \frac{d}{c}$ (which are distinct due to the hypothesis $ad - bc \neq 0$) are roots of $f - h$. Then any such integer quadratic polynomials must be of the form $t\,(ax - b)\,(cx - d)$ for some $t \in \mathbb{Z}$, because of the primitivity assumption $\gcd(a, b) = \gcd(c, d) = 1$. This establishes the inclusion $\mathcal{F} \subseteq \{h_t(x) : t \in \mathbb{Z}\}$. ■

Remark 3.4. As the lemma above shows, the set $\{h_t(x) : t \in \mathbb{Z}\}$ is exactly the set of cubic monic polynomials satisfying the conditions $a^3 f(\frac{b}{a}) = 1$ and $c^3 f(\frac{d}{c}) = 1$ appearing in Lemma 3.1. It is not true that any such polynomial is irreducible. For example,

$$f(x) = x^2(x - 2) + 1 = (x - 1)(x^2 - x - 1)$$

is not irreducible and yet it satisfies $1^3 f(\frac{0}{1}) = f(0) = 1$ and $1^3 f(\frac{2}{1}) = f(2) = 1$. In Theorem 3.12, we shall see that these $h_t(x)$ are irreducible for all but finitely many t, and then Lemma 3.1 will imply that the inclusion in the lemma above is cofinite. □

We note that at this point it is not obvious why $\mathcal{F} \neq \varnothing$. Below we will show that under suitable conditions on the parameters a, b, c, d, this is indeed the case and moreover, our conditions will imply that the units $a\theta - b, c\theta - d$ generate (together with -1) the group of units in $\mathbb{Z}[\theta]$.

Recall that by Dirichlet's theorem, in a totally real cubic field, the unit group modulo its torsion part has rank 2, and that two units are called a system of fundamental units if they generate the unit group modulo its torsion. If $a = 0$, then $0 \cdot \theta + b = b$ is a unit if and only if $b=\pm 1$, but of course it will not be a part of a system of fundamental units, and therefore we must have that $a\neq 0$, and similarly $c\neq 0$. On the other hand, if $b=0$, then $a\theta$ can be a unit only when $a=\pm 1$, namely θ is a unit. If d is also zero, then we get two units in $\{\pm\theta\}$ which cannot be a fundamental system. We therefore assume in our discussion $a,c\neq 0$ and at least one of b,d is non-zero.

3.2 The case where $b=0$ or $d=0$

We analyze the case $d=0$, that is, $a\theta-b,\theta$ form a fundamental set of units and the case $b=0$ is symmetric. We prove the following.

Theorem 3.5. Let $a,b\in\mathbb{Z}\backslash\{0\}$ such that $\gcd(a,b)=1$. There exists a monic cubic polynomial $f(x)\in\mathbb{Z}[x]$ such that $a^3 f\left(\frac{b}{a}\right)=1$ and $f(0)=1$ if and only if $a^3\equiv_b 1$ and $b^3\equiv_a 1$. In this case, there are infinitely many polynomials that satisfy this condition and they have the form

$$f_{a,b,t}(x)=\left(x^3+\frac{(a^3-1)^2-b^3}{ab^2}x^2-a(\frac{a^3-1}{b})x+1\right)+t\cdot x(ax-b),\tag{1}$$

where $t\in\mathbb{Z}$. Although this polynomial is supposed to be denoted by $f_{a,b,1,0,t}$, we omit the fixed parameters from the subscript to ease the notation. In particular, if f is irreducible and $\theta:=\theta_{a,b,t}$ is a root of $f_{a,b,t}$, then $\theta,a\theta-b$ are units in $\mathbb{Z}[\theta]$. □

Proof. The last sentence in the statement of the theorem follows directly from Lemma 3.1.

Let $f(x)=x^3+Ax^2+Bx+1$ with $A,B\in\mathbb{Z}$ be a generic monic cubic polynomial with integral coefficients and $f(0)=1$.

The condition $1=a^3 f\left(\frac{b}{a}\right)=b^3+Ab^2a+Ba^2b+a^3$ is equivalent to finding a solution for $A=\frac{1-b^3-a^3-Ba^2b}{ab^2}$, where $A,B\in\mathbb{Z}$, which immediately implies that $a^3\equiv_b 1$ and $b^3\equiv_a 1$.

On the other hand, if these congruence conditions hold, then using the fact that $gcd(a,b)=1$ we get that there is a solution with $A,B\in\mathbb{Z}$ if and only if

$$0\equiv_{b^2} 1-a^3-Ba^2b=b(\frac{1-a^3}{b}-Ba^2)$$
$$0\equiv_b \frac{1-a^3}{b}-Ba^2.$$

Multiplying the last expression by a, we get that $B = a\frac{1-a^3}{b} - bt$ for some $t \in \mathbb{Z}$ and therefore

$$A = \frac{1 - b^3 - a^3 - (a\frac{1-a^3}{b} - bt)a^2 b}{ab^2} = \frac{(1-a^3)^2 - b^3}{ab^2} + ta,$$

which completes the proof. ■

Example 3.6. In this example, we work out a simple recipe and show how to construct an infinite family of mutually cubic root pairs (a, b) (i.e., $a^3 \equiv_b 1$ and $b^3 \equiv_a 1$).

- The pairs $(a, 1), (1, b)$ are always mutually cubic root pairs. In these cases, the polynomials are

$$f_{1,b,t}(x) = x^3 - bx^2 + 1 + tx(x-b) = x(x+t)(x-b) + 1$$
$$f_{a,1,t}(x) = x^3 + \left[\left(a\left(a^3-1\right) - a + t\right)x - 1\right][ax-1]$$
$$= x^3 + [sx-1][ax-1] \quad ; \quad s = a\left(a^3-1\right) - a + t$$

- Given any b, in order to solve $0 \equiv_a b^3 - 1 = (b-1)\left(b^2+b+1\right)$ we can choose $a = b^2 + b + 1$. The second equation is satisfied automatically because $a^3 = \left(b^2+b+1\right)^3 \equiv_b 1^3 \equiv 1$.
- Similarly, given any b, we may take $a = 1-b$ and get that the two congruences are satisfied.
- Another option is to fix some integer r and set $a = r^2$ so we have $a^3 - 1 = \left(r^3\right)^2 - 1 = \left(r^3-1\right)\left(r^3+1\right)$. Thus, on choosing $b = r^3+1$ we get that $b^3 \equiv_a 1$ and the other congruence condition follows as well. □

We shall see in section 3.6 how to construct many more examples.

3.3 The case where both b, d are non-zero

We claim that if we wish $a\theta - b, c\theta - d$ to be independent units in $\mathbb{Z}[\theta]$ we need to assume $ad - bc \neq 0$. Otherwise we would have that $c\theta - d = \frac{ad}{b}\theta - d = \frac{d}{b}(a\theta - b)$, and because $N(a\theta - b), N(c\theta - d) = \pm 1$ we would get that $\frac{d}{b} = \pm 1$. It follows that $a\theta - b = \pm(c\theta - d)$, namely these units are not independent.

To make life easier, we will assume further that $ad - bc = 1$ (the case of $ad - bc = -1$ will follows from switching between a and b and switching between c and d). We prove the following analogue of Theorem 3.5.

Theorem 3.7. Let $a, b, c, d \in \mathbb{Z}\setminus\{0\}$ such that $ad - bc = 1$. Then there exists a monic cubic polynomial $f(x) \in \mathbb{Z}[x]$ such that $a^3 f\left(\frac{b}{a}\right) = 1$ and $c^3 f\left(\frac{d}{c}\right) = 1$ if and only if $b^3 \equiv_a 1$ and $d^3 \equiv_c 1$.

In this case there are infinitely many monic cubic polynomials that satisfy this condition and they have the form $f_{a,b,c,d,t}(x) = x^3 + Px^2 + Qx + R$ with $t \in \mathbb{Z}$ where

$$R = d^3 - b^3 + tbd$$
$$(P, Q) = \left(\frac{1 - b^3 - Ra^3}{ab}, \frac{1 - d^3 - Rc^3}{cd}\right)\begin{pmatrix} c & -d \\ -a & b \end{pmatrix}.$$

In particular, if $f_{a,b,c,d,t}(x)$ is irreducible and $\theta := \theta_{a,b,c,d,t}$ is its root, then $a\theta - b, c\theta - d$ are units in $\mathbb{Z}[\theta]$. □

Remark 3.8. Note that although it is not apparent by the formula above, due to Lemma 3.3, the cubic polynomials arising in the above theorem are all of the form $h_0(x) + tg(x)$, where $g = (ax - b)(cx - d)$. □

Proof. The last sentence in the statement of the theorem, regarding the units of $\mathbb{Z}[\theta]$, follows directly from Lemma 3.1.

For the main part of the theorem, we note first that $ad - bc = 1$ implies that $\gcd(a, b) = \gcd(a, c) = \gcd(d, b) = \gcd(d, c) = 1$.

Suppose that $f(x)$ is a monic cubic polynomial such that $a^3 f\left(\frac{b}{a}\right) = 1$ and $c^3 f\left(\frac{d}{c}\right) = 1$. Writing $f(x) = x^3 + Px^2 + Qx + R$, we need to satisfy the equations

$$1 = b^3 + Pb^2 a + Qba^2 + Ra^3 \quad \Longleftrightarrow \quad ab(Pb + Qa) = 1 - b^3 - Ra^3$$
$$1 = d^3 + Pd^2 c + Qdc^2 + Rc^3 \quad \Longleftrightarrow \quad cd(Pd + Qc) = 1 - d^3 - Rc^3.$$

We conclude that $ab \mid 1 - b^3 - Ra^3$, and since a, b are coprime, this condition is equivalent to $b^3 \equiv_a 1$ and $Ra^3 \equiv_b 1$. Similarly we get that $d^3 \equiv_c 1$ and $Rc^3 \equiv_d 1$, thus proving the first direction of the theorem.

Assume now that $b^3 \equiv_a 1$ and $d^3 \equiv_c 1$. Since $(a, b) = (c, d) = 1$, there are solutions to $Aa^3 \equiv_b 1$ and $Cc^3 \equiv_d 1$. Using $(b, d) = 1$ and the Chinese remainder theorem, we conclude that there is a solution $R \equiv_b A$ and $R \equiv_d C$ so that $Ra^3 \equiv_b 1$ and $Rc^3 \equiv_d 1$ and it is unique modulo bd. Once we have such an R we get that

$$(P, Q)\begin{pmatrix} b & d \\ a & c \end{pmatrix} = \left(\frac{1 - b^3 - Ra^3}{ab}, \frac{1 - d^3 - Rc^3}{cd}\right)$$

$$(P,Q)=\left(\frac{1-b^3-Ra^3}{ab},\frac{1-d^3-Rc^3}{cd}\right)\begin{pmatrix} c & -d \\ -a & b \end{pmatrix}$$

so that P,Q are also integers, thus completing the first part of the theorem, namely there exists a monic cubic polynomial $f(x)$ such that $a^3 f\left(\frac{b}{a}\right)=1$ and $c^3 f\left(\frac{d}{c}\right)=1$.

Assume now that we have a solution to the above equations. By our assumption, we have that

$$R\equiv_b R(bc+1)^3=R(ad)^3\equiv_b d^3$$
$$R\equiv_d -R(ad-1)^3=-R(bc)^3\equiv_d -b^3.$$

Since $(b,d)=1$, the Chinese remainder theorem implies that all the solutions have the form $R=d^3-b^3+tbd$ which completes the proof. ■

Example 3.9. We choose (a,b) as in the second bullet of Example 3.6, that is, $(a,b):=\left(b^2+b+1,b\right)$. To find (c,d) which solve the equation $ad-bc=1$ we choose for example $(c,d)=(b+1,1)$ and note that $d^3\equiv_c 1^3=1$ so that the conditions of the theorem are satisfied. □

The next lemma generalizes the example above and using the results from section 3.6 it produces infinitely many examples of suitable tuples a,b,c,d for the theorem above.

Lemma 3.10. Let (a,c) be a pair such that $a^3\equiv_c 1$ and $c^3\equiv_a -1$. Then $\gcd(a,c)=1$ and the integers b,d such that $ad-bc=1$ satisfy $b^3\equiv_a 1$ and $d^3\equiv_c 1$. □

Proof. Since $ad-bc=1$ we get that

$$b^3\equiv_a -(bc)^3=-(ad-1)^3\equiv_a 1$$
$$d^3\equiv_c (ad)^3=(1+bc)^3\equiv_c 1.$$

■

3.4 Full escape of mass

Fix some integers a,b,c,d which satisfy the conditions in Theorem 3.5 or 3.7. Our goal in this section is to show that if θ_t is a root of $h_t\stackrel{\text{def}}{=}f_{a,b,c,d,t}(x)$, then the mass of the orbits $A\cdot L_{\mathbb{Z}[\theta_t]}$ which corresponds to the orders $\mathbb{Z}[\theta_t]$ escape to infinity as $|t|\to\infty$. Moreover, we shall show that the Dirichlet shapes of the unit lattices in this family always converge to the regular triangles lattice.

For that, we will need to find good approximations for the roots of h_t. Heuristically, as $h_t = h_0 + tg$ with $g = (ax - b)(cx - d)$, when t is large, h_t will have roots close to the roots $\frac{b}{a}, \frac{d}{c}$ of g, and as it is cubic, its third root will also be real. This simple idea is developed further below. We shall use the following procedure. Since $f_{a,b,c,d,t}\left(\frac{b}{a}\right) = \pm\frac{1}{a^3}$ we will start with a guess that $\frac{b}{a}$ is close to the root. We will then use Taylor expansion and the Newton Raphson method to approximate the root.

Theorem 3.11. Let $h_t(x)$ be a familiy of polynomials and $\alpha_t \in \mathbb{R}$. Assume that

(1) $h_t'(\alpha_t) \neq 0$.
(2) $\lim_{t\to\infty} \left|\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right| = 0$.
(3) $\lim_{t\to\infty} \left|\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right| \left|\frac{h_t''(\alpha_t+\lambda)}{h_t'(\alpha_t)}\right| = 0$ uniformly in $|\lambda| \leq 1$. In particular this will be true if $\left|\frac{h_t''(\alpha_t+\lambda)}{h_t'(\alpha_t)}\right|$ is uniformly bounded (in t and $|\lambda| \leq 1$).

Then for t large enough the h_t have roots θ_t which satisfy

$$\theta_t = \alpha_t - \frac{h_t(\alpha_t)}{h_t'(\alpha_t)} + o\left(\left|\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right|\right). \qquad \square$$

Proof. We start by getting a first approximation for the root. Letting $\varepsilon = -2\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}$ and using the Taylor expansion for h_t we get that for some $|\lambda| \leq 1$ we have

$$\begin{aligned} h_t(\alpha_t + \varepsilon) &= h_t(\alpha_t) + h_t'(\alpha_t)\varepsilon + \frac{h_t''(\alpha_t + \lambda\varepsilon)}{2}\varepsilon^2 \\ &= h_t(\alpha_t)\left[-1 + 2\frac{h_t''(\alpha_t + \lambda\varepsilon)}{h_t'(\alpha_t)}\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right]. \end{aligned}$$

For t big enough $|\varepsilon| \leq 1$ so that $|\varepsilon\lambda| \leq 1$ hence we can use assumption (3) to also assume that the term in the brackets is negative. We conclude that $h_t(\alpha_t + \varepsilon), h_t(\alpha_t)$ have opposite signs and therefore h_t has a root $\theta_t \in [\alpha_t, \alpha_t + \varepsilon]$.

Applying the taylor expansion for θ_t and using $|\alpha_t - \theta_t| \leq |\varepsilon| = 2\left|\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right|$ we get

$$\begin{aligned} 0 &= h_t(\alpha_t) + h_t'(\alpha_t)(\theta_t - \alpha_t) + \frac{h_t''(\alpha_t + \lambda\varepsilon)}{2}(\theta_t - \alpha_t)^2 \\ \left|(\theta_t - \alpha_t) + \frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right| &= \left|\frac{h_t''(\alpha_t + \lambda\varepsilon)}{2h_t'(\alpha_t)}\right| |\theta_t - \alpha_t|^2 \leq 4\left|\frac{h_t''(\alpha_t + \lambda\varepsilon)}{2h_t'(\alpha_t)}\right| \left|\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right|^2 \\ &= 2\left|\frac{h_t''(\alpha_t + \lambda\varepsilon)h_t(\alpha)}{h_t'(\alpha_t)h_t'(\alpha_t)}\right| \left|\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right| = o\left(\left|\frac{h_t(\alpha_t)}{h_t'(\alpha_t)}\right|\right) \end{aligned}$$

and we are done. ■

We now consider the case where a, b, c, d are fixed and t goes to infinity. In what follows, we use the notation $f(t) \in \Theta(g(t))$ if there exists some constant $C > 0$ such that $\frac{1}{C} < \frac{f(t)}{g(t)} < C$ for all $|t|$ big enough.

Theorem 3.12. Fix $a, b, c, d \in \mathbb{Z}$ such that $\frac{b}{a} \neq \frac{d}{c}$, $a, c \neq 0$ and there exists a monic cubic polynomial $h(x)$ satisfying $a^3 h\left(\frac{b}{a}\right) = 1,\ c^3 h\left(\frac{d}{c}\right) = 1$. If $\frac{b}{a} + \frac{d}{c} \in \mathbb{Z}$, we will further assume that $a \neq -c$. We denote $h_t(x) = h(x) + tg(x), \quad g(x) = (ax - b)(cx - d)$, where $t \in \mathbb{Z}$. Then the following holds

(1) For $|t|$ big enough the polynomial $h_t(x)$ is totally real and irreducible.

(2) The 3 roots of $h_t(x)$ satisfy

$$\theta_1 = \frac{b}{a} - \Theta\left(\frac{1}{a^3 (bc - ad) t}\right)$$
$$\theta_2 = \frac{d}{c} + \Theta\left(\frac{1}{c^3 (bc - ad) t}\right)$$
$$\theta_3 = -act + O(1)$$

(3) The discriminant of h_t is

$$D_{h_t} = (\theta_1 - \theta_2)^2 (\theta_2 - \theta_3)^2 (\theta_3 - \theta_1)^2 = \left(\frac{b}{a} - \frac{d}{c}\right)^2 (act)^4 + O\left(t^3\right) \qquad \square$$

Proof. We first note that since a, b, c, d are fixed we get that $h_t(x) = x^3 + P_t x^2 + Q_t x + R_t$ and

$$P_t = act + O(1)$$
$$Q_t = -(ad + bc)\, t + O(1)$$
$$R_t = bdt + O(1).$$

Using these approximations and the hypothesis we get that

$$h_t\left(\frac{b}{a}\right) = \frac{1}{a^3}, \quad h_t'\left(\frac{b}{a}\right) = 2P_t \frac{b}{a} + Q_t + O(1) = (bc - ad)\, t + O(1)$$
$$h_t''\left(\frac{b}{c} + \lambda\right) = 2P_t + O(1) = 2act + O(1) \quad \forall\, |\lambda| \leq 1$$

It is now clear that

$$\frac{h_t(b/a)}{h_t'(b/a)} = \frac{1/a^3}{(bc - ad)\, t + O(1)} = \Theta\left(\frac{1}{a^3 (bc - ad)\, t}\right) \to 0$$

$$\frac{h_t''(b/a+\lambda)}{h_t'(b/a)} = \frac{2act + O(1)}{(bc-ad)\,t + O(1)} \to \frac{2ac}{bc-ad}$$

Hence, we can use Theorem 3.11 to approximate the root near $\frac{b}{a}$ and similarly the roots near $\frac{d}{c}$ which are

$$\theta_1 = \frac{b}{a} - \frac{h_t(b/a)}{h_t'(b/a)} + o\left(\frac{h_t(b/a)}{h_t'(b/a)}\right)$$
$$\theta_2 = \frac{d}{c} - \frac{h_t(d/c)}{h_t'(d/c)} + o\left(\frac{h_t(d/c)}{h_t'(d/c)}\right).$$

Note that since $\frac{b}{a} \neq \frac{d}{c}$, these two roots are distinct for $|t|$ big enough, so that $h_t(x)$, which is real of degree three, has at least two real roots, and therefore has exactly three real roots.

We claim that for $|t|$ big enough, the roots of $h_t(x)$ are not integers. If $\frac{b}{a} \notin \mathbb{Z}$, then for $|t|$ big enough we see that $\theta_1 \notin \mathbb{Z}$. If $\frac{b}{a} \in \mathbb{Z}$, then for $|t|$ big enough θ_1 can be an integer if and only if it is $\frac{b}{a}$ but $h_t(\frac{b}{a}) = \pm\frac{1}{a^3} \neq 0$. It follows that $\theta_1, \theta_2 \notin \mathbb{Z}$ for $|t|$ big enough. Finally, since $\theta_3 = -P_t - \theta_1 - \theta_2$ and P_t is an integer, we see that θ_3 is an integer if and only if $\theta_1 + \theta_2$ is an integer. If $\frac{b}{a} + \frac{d}{c} \notin \mathbb{Z}$, then θ_3 is not an integer for $|t|$ large enough. If $\frac{b}{a} + \frac{d}{c} \in \mathbb{Z}$, then we need to consider the second approximation

$$\frac{h(b/a)}{h'(b/a) + tg'(b/a)} + \frac{h(d/c)}{h'(d/c) + tg'(d/c)}$$
$$= \frac{1}{act}\left[\frac{1}{a^3}\frac{1}{\frac{h'(b/a)}{act} + \left(\frac{b}{a} - \frac{d}{c}\right)} + \frac{1}{c^3}\frac{1}{\frac{h'(d/c)}{act} + \left(\frac{d}{c} - \frac{b}{a}\right)}\right]$$

The limit of the expression inside the brackets is $\frac{1}{\left(\frac{b}{a}-\frac{d}{c}\right)}\left[\frac{1}{a^3} + \frac{1}{c^3}\right]$. We assumed that in this case $a \neq -c$ so that the root θ_3 modulo $\mathbb{Z}$ is $\Theta\left(\frac{1}{t}\right) + o\left(\frac{1}{t}\right)$ and in particular it is not an integer.

We showed that $h_t(x)$ does not have integer roots for $|t|$ big enough, and since it monic and has degree 3, we conclude that it is irreducible by using Gauss' lemma.

Finally, the approximation of the discriminant follows from the approximation of the roots. ■

Recall from Corollary 2.2 that if $\frac{R'}{\log^2(D)} < \frac{1}{8}$, where R' is the relative discriminant for some independent units, then these units are actually a fundamental set. In the following theorem, we will use this in order to show that $a\theta - b,\ c\theta - d$ are fundamental for $|t|$ big enough.

Theorem 3.13. Consider a family of polynomials $h_t(x)$ as in Theorem 3.12 and choose a root $\theta^{(t)}$ for each t. Then for t large enough the unit group of $\mathbb{Z}[\theta^{(t)}]$ is generated by $\left\{a\theta^{(t)}-b, c\theta^{(t)}-d, -1\right\}$. Furthermore, the Dirichlet shape of unit lattices of $\mathbb{Z}\left[\theta^{(t)}\right]$ converge to the shape of the regular triangle lattice $\mathbb{Z}[\omega]$ (where $\omega = \exp(\frac{2\pi i}{3})$ and the correspondence is $\left(a\theta^{(t)}-b\right) \mapsto 1$ and $\left(c\theta^{(t)}-d\right) \mapsto (1+\omega)$) and the compact A-orbits of the lattices $L_t \in X$ corresponding to the orders $\mathbb{Z}[\theta^{(t)}]$ exhibit full escape of mass. □

Proof. The embedding of the units $a\theta^{(t)}-b,\ c\theta^{(t)}-d$ in $\mathbb{R}^3$ is

$$\begin{aligned}
&\log\left(\left|a\theta^{(t)}-b\right|\right)\\
&=\left(-\log\left|a^2(bc-ad)t\right|, \log\left|\frac{ad}{c}-b\right|, \log\left|a^2ct\right|\right)+O(1)\\
&=\log|t|(-1,0,1)+O(1)\\
&\log\left(\left|c\theta^{(t)}-d\right|\right)\\
&=\left(\log\left|\frac{cb}{a}-d\right|, -\log\left|c^2(bc-ad)t\right|, \log\left|ac^2t\right|\right)+O(1)\\
&=\log|t|(0,-1,1)+O(1)
\end{aligned}$$

The relative regulator is $\log^2|t| + O(\log|t|)$ so that

$$\frac{R'_t}{\log^2(D_t)} = \frac{\log^2(t)+O(\log|t|)}{\left(\log\left|\left(\frac{b}{a}-\frac{d}{c}\right)^2(act)^4\right|+O(1)\right)^2} = \frac{\log^2(t)+O(\log|t|)}{(4\log|t|+O(1))^2} \to \frac{1}{16}.$$

It follows from Corollary 2.2 that for $|t|$ big enough, the units $a\theta^{(t)}-b,\ c\theta^{(t)}-d$ are a fundamental set.

We note that $(-1,0,1), (0,-1,1)$ generate the regular triangles lattice. Indeed, the rotation around $(1,1,1)$ by $\frac{2\pi}{3}$ is just the cyclic permutation, and these two vector are just the rotation of each other, up to a minus sign.

Using the simplex set

$$\Phi = \left\{\log|t|(-1,0,1), \log|t|(1,-1,0), \log|t|(0,1,-1)\right\}+O(1),$$

it is easily seen that $\lceil W_\Phi \rceil = \log|t|\frac{2}{3}+O(1)$. On the other hand $D_t^{-1/6}(1,1,1)$ is in the normalized unimodular lattice L_t that correspond to $\mathbb{Z}\left[\theta^{(t)}\right]$ so that $ht(L_t) \geq \frac{1}{\sqrt{3}}D_t^{1/6} = \Theta\left(t^{2/3}\right)$. We conclude that $\exp(\lceil W_\Phi \rceil) = O\left(t^{\frac{2}{3}}\right) \leq R \cdot ht(L_t)$ for some R big enough and all $|t|$ big enough, namely these orders are $(R,1)$-tight (see Definition 2.7 for the definitions).

It follows that there is a full escape of mass by Theorem 2.11 which completes the proof. ■

In the case of simplest cubic fields, the fundamental units are $\theta, \theta+1$. It is known that for infinitely many t, the order $\mathbb{Z}[\theta_t]$, where θ_t is the root of $f_t(x) = x^3-3x+1-tx(x+1)$, is the ring of integers of $\mathbb{Q}(\theta_t)$. In particular the $\mathbb{Z}[\theta_t]$ belong to different field extensions. We conclude that there are orbits coming from different fields such that their mass escape to infinity.

While we do not have an example of orbits arising from the same field, we can create long finite sequences of orbits such that most of their mass is near the cusp.

If the unit group is generated by $\langle na\theta - b, nc\theta - d\rangle$, then $\mathbb{Z}[\theta]$ and $\mathbb{Z}[n\theta]$ have the same unit group. Since $D_{n\theta} = n^6 D_\theta$, the mass of its corresponding orbit is farther away than the mass of $\mathbb{Z}[\theta]$. This leads to the following result.

Theorem 3.14. Let $1 > \varepsilon > 0$ and $K \subseteq SL_3(\mathbb{R})/SL_3(\mathbb{Z})$ be a compact set. Then for each $N \in \mathbb{N}$ we can find a sequence of decreasing orders $\mathbb{Z}[\theta_1] > \mathbb{Z}[\theta_2] > \cdots > \mathbb{Z}[\theta_N]$ with their corresponding orbits $A \cdot L_1, \ldots, A \cdot L_N$ such that $\frac{\mu_i(A\cdot L_i \cap K)}{\mu_i(A\cdot L_i)} < \varepsilon$ for each i where μ_i is the induced A-invariant measure on $A \cdot L_i$. □

Proof. Consider the polynomials $f_{t,n}(x) = x^3 + t(2^n x - 1)(2^{n-1}x - 1)$ with corresponding roots $\theta_{t,n}$. From Theorem 3.13, for a given compact set K and $\varepsilon > 0$ we can find T big enough such that for all $t > T$ the A-orbits $A \cdot L_{t,n}$ corresponding to the orders $\mathbb{Z}\left[\theta_{t,n}\right]$ satisfy $\frac{\mu_{t,n}(A\cdot L_{t,n} \cap K)}{\mu_{t,n}(A\cdot L_{t,n})} < \varepsilon$ for all $1 \le n \le N$.

Notice that the minimal polynomial for $2\theta_{t,n}$ is $f_{t,n}\left(\frac{x}{2}\right) = \frac{1}{8} f_{8t,n-1}$.

It follows that $2\theta_{t,n}$ is a root of $f_{8t,n-1}$. Since $\mathbb{Z}\left[2\theta_{t,n}\right] = span\left\{1, 2\theta_{t,n}, 4\theta_{t,n}^2\right\}$, we see that $\left[\mathbb{Z}\left[\theta_{t,n}\right] : \mathbb{Z}\left[2\theta_{t,n}\right]\right] = 8$, so these are distinct orders. Using induction, we get the orders $\mathbb{Z}\left[2^{N-1}\theta\right] < \mathbb{Z}\left[2^{N-2}\theta\right] < \cdots < \mathbb{Z}[2\theta] < \mathbb{Z}[\theta]$ where $\theta := \theta_{T,N}$, such that their corresponding orbits $A \cdot L_i$ all satisfy $\frac{\mu_i(A\cdot L_i \cap K)}{\mu_i(A\cdot L_i)} < \varepsilon$. ■

Problem 3.15. Is there an infinite sequence of lattices coming from a fixed field, or better yet, coming from a sequence of decreasing orders, which exhibits escape of mass? □

3.5 The lattices $\mathbb{Z}[\theta]$, where θ is a unit

In the previous section, the Dirichlet shapes of the unit lattices converged to the regular triangles lattice. In this section we show how to construct more examples with different unit lattice Dirichlet shape.

We shall now confine our attention to analyze sequences of polynomials arising from Theorem 3.5 where the parameters a, b are chosen as functions of t. More precisely, given a mutually cubic root sequence (a_t, b_t) we define (as in 1)

$$h_t(x) = f_{a_t,b_t,t}(x) = x^3 + P_t x^2 + Q_t x + 1 \tag{2}$$
$$(P_t, Q_t) = \left(\frac{\left(a^3-1\right)^2 - b^3}{ab^2}, -a\left(\frac{a^3-1}{b}\right)\right) + t(a, -b).$$

By Theorem 3.5, if θ_t is a root of h_t then $\theta_t, a_t\theta_t - b_t$ are units of the order $\mathbb{Z}[\theta_t]$. In fact, we will make the following standing assumption that will help us in the analysis.

Assumption 3.16. Let h_t be the sequence of polynomials in 2 corresponding to the mutually cubic root sequence (a_t, b_t) and assume furthermore that

(1) $|b_t| > |a_t| > 0$ for each t and
(2) $\tilde{a} := \lim_{t\to\infty} \frac{\log|a_t|}{\log(t)}$ and $\tilde{b} := \lim_{t\to\infty} \frac{\log|b_t|}{\log(t)}$ exist and $\tilde{a} < \frac{1}{3}$, $\tilde{b} < 1$. □

Remark 3.17. The assumption above implies that $a_t^r = o(t)$ for $r \leq 3$ and that $b_t = o(t)$, so that $P_t = at + o(t)$ and $Q_t = -bt + o(t)$. □

We remark that some of the claims below are true in a more general setting than the assumption above.

Theorem 3.18. Assume 3.16. Then the polynomial $h_t(x)$ is irreducible over $\mathbb{Q}$ for $|t|$ big enough. □

Proof. Since both the leading and free coefficient of h_t are ± 1, we get that h_t is reducible (over $\mathbb{Q}$) if and only if it has a root in ± 1.

$$h_t(\pm 1) = O(1) + P_t \pm Q_t = (a \pm b)\,t + o(t)$$

Since $b \neq \pm a$ are integers we conclude that $|h_t(\pm 1)| \geq \frac{t}{2}$ for $|t|$ big enough, and hence $h_t(\pm 1) \neq 0$. ■

Lemma 3.19. Assume 3.16. Then

(1) $\left|\frac{h_t''(\lambda)}{h_t'(0)}\right|$ is uniformly bounded for $|\lambda| \leq 1$ and $\left|\frac{h_t(0)}{h_t'(0)}\right| = \Theta\left(\frac{1}{|tb|}\right) \to 0$.
(2) $\left|\frac{h_t''(b/a+\lambda)}{h_t'(b/a)}\right|$ is uniformly bounded for $|\lambda| \leq 1$ and $\left|\frac{h_t(b/a)}{h_t'(b/a)}\right| = \Theta\left(\frac{1}{|a^3bt|}\right) \to 0$. □

Proof. (1) We have the following:

$$h_t(0) = 1, \quad \left|h_t'(0)\right| = |Q_t| = \Theta(|tb|)$$
$$|\lambda| \le 1 \quad \Rightarrow \quad \left|h_t''(\lambda)\right| = |6\lambda + 2P_t| = \Theta\left(|ta|\right)$$

We conclude that $\left|h_t''(\lambda)\right| = \Theta\left(|ta|\right) = O\left(\left|h_t'(0)\right|\right)$ uniformly over $|\lambda| \le 1$ since $|a| < |b|$, so that $\left|\frac{h_t''(\lambda)}{h_t'(0)}\right|$ is bounded. Since $|h_t(0)| = 1$ we have that $\left|\frac{h_t(0)}{h_t'(0)}\right| = \Theta\left(\frac{1}{|tb|}\right) \to 0$

(2) The second claim is similar — we have

$$h_t(\frac{b}{a}) = \frac{1}{a^3}, \quad \left|h_t'\left(\frac{b}{a}\right)\right| = \Theta\left(|bt|\right) \quad \Rightarrow \quad \left|\frac{h_t(b/a)}{h_t'(b/a)}\right| = \Theta\left(\frac{1}{|a^3bt|}\right) \to 0$$

Secondly, we have that

$$|\lambda| \le 1 \;\Rightarrow\; \left|h_t''(b/a + \lambda)\right| = \left|6\left(\frac{b}{a} + \lambda\right) + 2P_t\right| = \Theta(|ta|)$$
$$\left|\frac{h_t''\left(\frac{b}{a} + \lambda\right)}{h_t'\left(\frac{b}{a}\right)}\right| = \frac{\Theta\left(|ta|\right)}{\Theta\left(|tb|\right)} = O\left(\left|\frac{a}{b}\right|\right) = O\left(1\right),$$

so that the expression above is bounded. ■

Corollary 3.20. Assume 3.16. The roots and discriminant of $h_t(x)$ satisfy

$$\begin{aligned}
\theta_1 &= 0 - \frac{h_t(0)}{h_t'(0)} + o\left(\left|\frac{h_t(0)}{h_t'(0)}\right|\right) = \Theta\left(\frac{1}{|tb|}\right)\\
\theta_2 &= \frac{b}{a} - \frac{h_t(b/a)}{h_t'(b/a)} + o\left(\left|\frac{h_t(b/a)}{h_t'(b/a)}\right|\right) = \frac{b}{a}\left(1 + \Theta\left(\frac{1}{a^2b^2t}\right)\right)\\
\theta_3 &= \Theta\left(|at|\right)\\
D_f &= \Theta\left(t^4a^2b^2\right)
\end{aligned}$$
□

Proof. The approximations for θ_1, θ_2 follow from the previous lemma and Theorem 3.11. The third root satisfies

$$\theta_3 = -P_t - \theta_1 - \theta_2.$$

Note that $\theta_1 \to 0$ while $\theta_2 = \Theta\left(\frac{b}{a}\right) = o\left(t\right)$ so that $|\theta_3| = \Theta\left(|P_t|\right) = \Theta\left(|at|\right)$. Since $\theta_1 = o(\theta_2)$ and $\theta_2 = o(\theta_3)$ we get that the discriminant satisfies

$$D_t = \left(\theta_1 - \theta_2\right)^2\left(\theta_2 - \theta_3\right)^2\left(\theta_3 - \theta_1\right)^2 = \Theta\left(t^4a^2b^2\right).$$
■

We are now ready to show that $\theta, a\theta - b$ form a fundamental set, compute the Dirichlet shape of the unit lattice, and show that there is a partial escape of mass. Recall Definition 2.10 and Theorem 2.11.

Theorem 3.21. Assume 3.16. For h_t as above let θ_t be one of its roots and let $M_t = \mathbb{Z}[\theta_t]$, L_t be the corresponding order and unimodular lattice. Then for $|t|$ big enough we have:

(1) The units $\{\theta_t, a_t\theta_t - b_t\}$ are a set of fundamental units for M_t.

(2) There exist simplex sets for M_t which are (R, r)-tight for all $0 \le r < 1$ which satisfies $\frac{2}{3}(1-r) + \left(\frac{1}{3} - r\right)\left(\tilde{a} + \tilde{b}\right) > 0$. In particular, the family of compact A-orbits AL_t exhibits partial escape of mass (by choosing $r = \frac{1}{3}$) and there is a full escape if $\tilde{a}, \tilde{b} = 0$ (e.g., for a_t, b_t bounded). □

Proof. Recall that we embedd the units into $\mathbb{R}^3$ by sending a unit α to $(\log|\sigma_i(\alpha)|)_1^3$, where $\sigma_i : \mathbb{Z}[\theta_t] \to \mathbb{R}$ are the three real embeddings. Thus, using the previous corollary for approximating the units $\theta_t, a\theta_t - b$, we get that

$$(\log|\sigma_i(\theta_t)|)_1^3 = \left(-\log|tb|, \log\left|\frac{b}{a}\right|, \log|at|\right) + O(1)$$
$$(\log|a\sigma_i(\theta_t) - b|)_1^3 = \left(\log|b|, -\log\left|a^2bt\right|, \log\left|a^2t\right|\right) + O(1)$$

We prove (1). The relative regulator R_i' for these units is

$$\det\left(\begin{pmatrix} -\log|tb| & \log\left|\frac{b}{a}\right| \\ \log|b| & -\log\left|a^2bt\right| \end{pmatrix} + O(1)\right)$$
$$= \log|tb|\log\left|ta^2b\right| - \log|b|\log\left|\frac{b}{a}\right| + O(\log(t))$$

so that

$$\frac{R_i'}{\log^2(D_i)} = \frac{\log|tb|\log\left|ta^2b\right| - \log|b|\log\left|\frac{b}{a}\right| + O(\log(t))}{\log^2(t^4a^2b^2) + O(\log(t))}$$
$$\longrightarrow \frac{\left(1+\tilde{b}\right)\left(1+2\tilde{a}+\tilde{b}\right) - \tilde{b}\left(\tilde{b}-\tilde{a}\right)}{4\left(2+\tilde{a}+\tilde{b}\right)^2}.$$

We claim that for $0 \le \tilde{a}, \tilde{b} < 1$, the expression above is always smaller than $\frac{1}{8}$, and therefore the units $\theta_t, a\theta_t - b$ form a fundamental set of units (see Corollary 2.2). Indeed, the expression is strictly less than $\frac{1}{8}$ if and only if

$$0 \overset{?}{\leq} 4\left(2+\tilde{a}+\tilde{b}\right)^2 - 8\left[\left(1+\tilde{b}\right)\left(1+2\tilde{a}+\tilde{b}\right) - \tilde{b}\left(\tilde{b}-\tilde{a}\right)\right]$$
$$= 8 + 4\tilde{a}^2 + 4\tilde{b}^2 - 16\tilde{a}\tilde{b} = 4\left(\tilde{b}-\tilde{a}\right)^2 + 8\left(1-\tilde{a}\tilde{b}\right)$$

This is clearly true if $0 \leq \tilde{a}, \tilde{b} \leq 1$ and the equality holds only if $\tilde{a} = \tilde{b} = 1$.

We prove (2). Instead of working with $\theta_t, a\theta_t - b$, we shall work with the simplex set $\Phi = \left\{\theta_t, \theta_t^{-1}(a\theta_t - b), (a\theta_t - b)^{-1}\right\}$. These units correspond to

$$\left(\log\left|\sigma_i\left(\theta^{-1}(a\theta-b)\right)\right|\right)_1^3$$
$$= -\left(-\log|tb|,\ \log\left|\frac{b}{a}\right|,\ \log|at|\right) + \left(\log|b|, \log\left|\frac{1}{a^2bt}\right|, \log\left|a^2t\right|\right) + O(1)$$
$$= \left(\log\left|tb^2\right|, -\log\left|ab^2t\right|, \log|a|\right) + O(1).$$
$$\left(\log\left|\sigma_i(a\theta-b)^{-1}\right|\right)_1^3 = \left(-\log|b|, \log\left|a^2bt\right|, -\log\left|a^2t\right|\right) + O(1).$$

The vertices of the fundamental domain $\mathrm{conv}(W_\Phi)$ correspond to

$$\theta^{\lambda_1}\left(a - b\theta^{-1}\right)^{\lambda_2}\left((a\theta-b)^{-1}\right)^{\lambda_3},$$

where $\{\lambda_1, \lambda_2, \lambda_3\} = \left\{0, \frac{1}{3}, \frac{2}{3}\right\}$ (see Definition 2.7). From these vertices we need to find the maximum of the coordinates. For example, on the first coordinate we have $-\log|tb|,\ \log\left|tb^2\right|,\ -\log|b|$. To get a maximum, we clearly need to assign the $\frac{2}{3}$ power to $\log\left|tb^2\right|$ (which is positive) and $\frac{1}{3}$ to $-\log|b|$ (which is bigger than $-\log|tb|$), hence obtaining $\log\left(\left|tb^2\right|^{2/3} \cdot |b|^{-1/3}\right) = \log\left(|t|^{\frac{2}{3}}|b|\right)$. A similar computation for the second and third coordinate will produce $\log\left(\left|a^2bt\right|^{\frac{2}{3}}\left|\frac{b}{a}\right|^{\frac{1}{3}}\right) = \log\left(|t|^{\frac{2}{3}}|ab|\right)$ and $\log\left(|at|^{\frac{2}{3}}|a|^{\frac{1}{3}}\right) = \log\left(|t|^{\frac{2}{3}}|a|\right)$. It follows that the maximum is $\lceil W_\Phi \rceil = \log\left(|t|^{\frac{2}{3}}|ab|\right) + O(1)$.
The height of the unimodular lattice is controlled by the size of $D^{-1/6}(1,1,1)$, so that $ht(L_i) = \Theta\left(t^{\frac{2}{3}}|ab|^{\frac{1}{3}}\right)$. The (R, r)-tightness condition is

$$\exp\left(r\left\lceil\tilde{\Phi}\right\rceil\right) \leq R \cdot ht(L_i) \quad \Longleftrightarrow \quad -\log(R) \leq \log|ht(L_i)| - r\left\lceil\tilde{\Phi}\right\rceil$$

so it is enough to show that $\log|ht(L_i)| - r\left\lceil\tilde{\Phi}\right\rceil \to \infty$.

$$\log|ht(L_i)| - r\left\lceil\tilde{\Phi}\right\rceil = \log\left(t^{\frac{2}{3}}|ab|^{\frac{1}{3}}\right) - r\log\left(|t|^{\frac{2}{3}}|ab|\right) + O(1)$$
$$= \log|t|\left(\frac{2}{3}(1-r) + \left(\frac{1}{3} - r\right)\frac{\log|ab|}{\log|t|}\right) + O(1)$$

Thus, by taking $|t| \to \infty$, we see that the condition is equivalent to

$$\frac{2}{3}(1-r) + \left(\frac{1}{3} - r\right)\left(\tilde{a} + \tilde{b}\right) > 0.$$

We immediately see that if $r \le \frac{1}{3}$, then this condition is always satisfied and therefore we always have partial escape of mass. On the other extreme, if $\tilde{a}, \tilde{b} = 0$ (e.g., if a_t, b_t are bounded), then the inequality is true for all $r < 1$, so that we have a full escape of mass. ■

Theorem 3.22. Assume 3.16. For h_t as above let θ_t be one of its roots and let $M_t = \mathbb{Z}[\theta_t]$, L_t be the corresponding order and unimodular lattice. Let $[z_t] \in SL_2(\mathbb{Z})\backslash\mathbb{H}$ be the Dirichlet shape of the unit lattice $\psi\left(\mathbb{Z}[\theta_t]^\times\right) \subset \mathbb{R}^3_0$. Then the sequence $[z_t]$ converges to some point $[z] \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ where $z \in \mathbb{H}$ satisfies the following:

- $z(\tilde{a}, \tilde{b}) = \frac{1+2\tilde{a}+\left(1+\tilde{b}+2\tilde{a}\right)\omega}{1+\tilde{a}+\left(\tilde{a}-\tilde{b}\right)\omega}$, where $\omega = \frac{-1+\sqrt{3}i}{2}$ is a primitive root of unity of order 3.
- If $\tilde{a} = \tilde{b} = 0$, then $z = 1 + \omega$, or equivalently the lattice shape is the regular triangles lattice.
- If $\tilde{a} = 0$ then $|z| = 1$.
- If $\tilde{a} = \tilde{b}$, then $Re(z) = \frac{1}{2}$.

Moreover, if $0 < \tilde{a} < \tilde{b}$ are small enough (say, $< \frac{1}{10}$), then z is in the interior of the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathbb{H}$. □

Proof. From Theorem 3.21, the unit lattice is generated by the elements

$$\begin{aligned}
&\begin{pmatrix} \log|\sigma_i(\theta_t)| \\ \log|a\sigma_i(\theta_t)-b| \end{pmatrix}_1^3 \\
&= \log(t)\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} + \log|b|\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + \log|a|\begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix} + O(1) \\
&= \log(t)\left[\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{\log|b|}{\log|t|}\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + \frac{\log|a|}{\log|t|}\begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix} + O\left(\frac{1}{\log|t|}\right)\right].
\end{aligned}$$

The vectors $\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}$ have the same norm and have angle $\frac{\pi}{3}$ between them, so we can define a similarity from $\mathbb{R}^3_0$ to $\mathbb{C}$ by sending them to $1, 1+\omega$, respectively. We thus get a lattice in $\mathbb{C}$ having the same shape which is generated (in the limit as $|t| \to \infty$) by

$$v = 1 - \tilde{b}\omega + \tilde{a}(1+\omega) = 1 + \tilde{a} - \left(\tilde{b} - \tilde{a}\right)\omega$$

$$u = 1 + \omega + \tilde{b}\omega + 2\tilde{a}(1+\omega) = 1 + 2\tilde{a} + \left(1 + \tilde{b} + 2\tilde{a}\right)\omega.$$

This lattice has the same shape as the one generated by $1, \frac{u}{v}$, which is the claim in the first bullet of the theorem.

If $\tilde{a} = 0$, then $\frac{u}{v} = \frac{1+\left(1+\tilde{b}\right)\omega}{1-\tilde{b}\omega}$ and then $|\frac{u}{v}|^2 = \frac{1-\left(1+\tilde{b}\right)+\left(1+\tilde{b}\right)^2}{1+\tilde{b}+\tilde{b}^2} = 1$. Similarly, if $\tilde{a} = \tilde{b}$, then $\frac{u}{v} = \frac{1+2\tilde{a}+\left(1+3\tilde{a}\right)\omega}{1+\tilde{a}}$ so that

$$Re\left(\frac{u}{v}\right) = \left(\frac{1+2\tilde{a}}{1+\tilde{a}}\right) + \left(\frac{1+3\tilde{a}}{1+\tilde{a}}\right)\left(\frac{-1}{2}\right) = \frac{1}{2}.$$

We leave it as an exercise to show that for small $0 \le \tilde{a}, \tilde{b}$, the number $\frac{u}{v}$ is inside the standard fundamental domain and is strictly inside if $0 < \tilde{a} < \tilde{b}$. ■

We now have all we need in order to prove Theorem 1.4.

Proof of Theorem 1.4. Let (a_t, b_t) be a mutually cubic root sequence and suppose that the limits $\tilde{a} = \lim_{t\to\infty} \frac{\log|a_t|}{\log|t|}$ and $\tilde{b} = \lim_{t\to\infty} \frac{\log|b_t|}{\log|t|}$ exist and satisfy $0 \le \tilde{a} \le \tilde{b}$. Given $p, q \in \mathbb{N}$, we reindex the sequence (a_t, b_t, t) and consider the sequence (a_{t^p}, b_{t^p}, t^q). Obviously, this is also a mutually cubic root sequence, and the corresponding limits are $\frac{p}{q}\tilde{a}$ and $\frac{p}{q}\tilde{b}$. In particular, taking $r = \frac{p}{q} \in \left[0, \min(\frac{1}{3\tilde{a}}, \frac{1}{\tilde{b}})\right) \cap \mathbb{Q}$ we get a sequence that satisfies Assumption 3.16, hence we get the limit point $z(r\tilde{a}, r\tilde{b}) = \frac{1+2r\tilde{a}+\left(1+r\tilde{b}+2r\tilde{a}\right)\omega}{1+r\tilde{a}+\left(r\tilde{a}-r\tilde{b}\right)\omega}$ in $\overline{\Omega}$, where Ω was defined to be the set of Dirichlet shapes of unit lattices. Finally, using the continuity of $z(x, y)$ it follows that $z(r\tilde{a}, r\tilde{b}) \in \overline{\Omega}$ for all $r \in \left[0, \min(\frac{1}{3\tilde{a}}, \frac{1}{\tilde{b}})\right]$. ■

3.6 Finding the limits of $\lim_{t\to\infty} \frac{\log|a_t|}{\log|b_t|}$

As Theorem 1.4 shows, once we are given a mutually cubic root sequence (a_t, b_t), the parameter that controls the curve is the ratio $\frac{\tilde{a}}{\tilde{b}} = \lim_{t\to\infty} \frac{\log|a_t|}{\log|b_t|}$ and we are left with the task of finding such sequences inducing different ratios. Let us denote by Λ the set of such limits inside $P^1(\mathbb{R})$ (ignoring the case where $\tilde{a} = \tilde{b} = 0$), so that any $0 \le \lambda \le 1$ in Λ corresponds to a curve in $\overline{\Omega}$.

Before we turn to study the set Λ, let us concentrate on the case where $\tilde{a} = 0$. Since $b_t \mid a_t^3 - 1$, unless $a_t = 1$ for all t big enough, we will also get that $\tilde{b} \le 3\tilde{a} = 0$, which by Theorem 3.22 implies that the Dirichlet shapes of the unit lattices converge to the regular triangle lattices (which is like the case discussed in Section 3.4).

Assuming now that $a_t = 1$ for all t, Theorem 3.5 tell us that

$$f_t(x) = \left(x^3 - b_t x^2 + 1\right) + t \cdot x\,(x - b_t) = x\,(x - b_t)\,(x + t) + 1.$$

This type of polynomials was already studied by Cusick in [2] where he showed that the limit points of the Dirichlet shapes of unit lattices is on $|z| = 1$ in the hyperbolic plane. This follows readily from our computations in the previous section if $b_t = o(t)$ and in addition we know that there is always a partial escape of mass. Moreover, when $b_t \sim t^\alpha$, $\alpha < 1$, the cosine of the angle of the corresponding point on the hyperbolic plane is $\frac{1-2\alpha-2\alpha^2}{2+2\alpha+2\alpha^2}$. In particular, the angle is $\frac{\pi}{3}$ when $\alpha = 0$ and it increases up until $\frac{2\pi}{3}$ when $\alpha \to 1^-$. In case that $b_t = Bt$ for some constant B, our analysis does not hold. This case falls into the settings studied in [10] where it was shown that there is a full escape of mass and that the Dirichlet shapes of the unit lattices converge to the regular triangles lattice.

Let us continue to the general case of an element in Λ. We have already seen several examples in Example 3.6 of mutually cubic root sequences producing the limits $0, \infty, \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \in \Lambda$, so that Λ is not empty. On the other hand, if $a_t, b_t \neq \pm 1$ and $b_t^3 \equiv_{a_t} 1$, then $|a_t| \mid \left|b_t^3 - 1\right|$ so that $\lim\limits_{t\to\infty} \frac{\log|a_t|}{\log|b_t|} \leq \lim\limits_{t\to\infty} \frac{\log\left|b_t^3-1\right|}{\log|b_t|} \to 3$ and reversal of the roles of a_t, b_t produces a lower bound $\frac{1}{3}$, so that $\Lambda \subseteq \left[\frac{1}{3}, 3\right] \cup \{0, \infty\}$.

Lemma 3.23. We have the following:

(1) The set Λ is closed under taking inverses.

(2) If $s \in \Lambda$, then $3 - s \in \Lambda$. □

Proof. (1) Clearly, any mutually cubic root sequence (a_t, b_t) produces another such sequence (b_t, a_t) so that $s \in \Lambda$ if and only if $s^{-1} \in \Lambda$. On the level of units, this is nothing more than considering the fundamental units $\left\{\theta^{-1}, -\theta^{-1}(a\theta - b)\right\}$ instead of $\{\theta, a\theta - b\}$.

(2) Let (a, b) be a mutually cubic roots pair and suppose first that $b \neq 1$. Setting $c = \frac{1-b^3}{a} \neq 0$ we get that $c \mid 1 - b^3$ so that $b^3 \equiv_c 1$. On the other hand, we have that

$$c^3 \equiv_b (ca)^3 = \left(1 - b^3\right)^3 \equiv_b 1$$

so that (c, b) is another mutually cubic root pair. Taking the limit we get that $\frac{\tilde{c}}{\tilde{b}} = \frac{3\tilde{b}-\tilde{a}}{\tilde{b}} = 3 - \frac{\tilde{a}}{\tilde{b}}$.

If on the other hand $b_t = 1$ for almost all t, then $\frac{\tilde{a}}{\tilde{b}} = \infty$ (since we assumed that $(\tilde{a}, \tilde{b}) \neq (0, 0)$), and then $3 - \infty = \infty \in \Lambda$, hence the claim is still true. ■

Both of the maps $s \to s^{-1}$ and $s \to 3 - s$ have order 2, but their composition $T(s) = 3 - \frac{1}{s}$ has infinite order and acts on $P^1(\mathbb{R})$. We start with some basic properties of this Möbius action.

Lemma 3.24. Let $T(s) = 3 - \frac{1}{s}$. Then:

(1) The fixed points of T are $\alpha_{\pm} = \frac{3\pm\sqrt{5}}{2}$ where $\frac{1}{3} < \alpha_{-} < \alpha_{+} < 3$. In addition, any other T orbit is infinite.

(2) $T[(\alpha_{-}, \alpha_{+})] = (\alpha_{-}, \alpha_{+})$ and $T(s) > s$ in this segment.

(3) $T[(\alpha_{+}, \infty]] = (\alpha_{+}, 3]$ and $T(s) < s$ for $s \in (\alpha_{+}, \infty]$.

(4) $T^{-1}[[0, \alpha_{-})] = [\frac{1}{3}, \alpha_{-})$ and $T(s) < s$ for $s \in [\frac{1}{3}, \alpha_{-})$.

(5) T satisfies $T(\frac{1}{3}) = 0$, $T(0) = \infty$ and $T(\infty) = 3$.

(6) The only accumulation points of a single T-orbit in Λ are $\alpha_{\pm}$. □

Proof. Left as an exercise. ■

Corollary 3.25. Λ is infinite. □

Proof. Since $3, \frac{1}{3}, \frac{1}{2} \in \Lambda$, application of Lemma 3.24 provides 2 infinite T-orbits in Λ, and in particular Λ is infinite in itself. ■

It is now easily seen that Λ contains at least two accumulation points at $\frac{3\pm\sqrt{5}}{2}$ which are exactly the fixed points of T. Define $\tilde{T}$ to be the corresponding action on sequences of mutually cubic roots, that is, $\tilde{T}(a_t, b_t) = (\frac{1-a_t^3}{b_t}, a_t)$. This is a composition of switching the sequences $(a_t, b_t) \mapsto (b_t, a_t)$ and then using the fact that $b_t^3 - 1 \equiv_{a_t} 0$ we map $(b_t, a_t) \mapsto (\frac{1-a_t^3}{b_t}, a_t)$.

Suppose now that $b_t \mid a_t^2 + a_t + 1$, for example, the sequence $(t, 1)$. In this case we will get that $(1-a_t)b_t \mid a_t^3 - 1$ so we may define the operation $\tilde{D}(a_t, b_t) = (a_t, (1-a_t)b_t)$ and hope to get a new mutually cubic root pair. As the next lemma shows, while we cannot use this operation on any sequence of mutually cubic root pairs, there are enough such sequences.

Lemma 3.26. Define $\tilde{D} : (\mathbb{Z} - \{1\}) \times \mathbb{Z} \to (\mathbb{Z} - \{1\}) \times \mathbb{Z}$ by $D(a_t, b_t) = (a_t, (1 - a_t)b_t)$. Then the following holds:

(1) The map $\tilde{D}$ induces a bijection between the set of mutually cubic root pairs (a, b) with $b \mid a^2 + a + 1$ and the set of mutually cubic root pairs (c, d) with $(1 - c) \mid d$.

(2) Suppose that (a_t, b_t) is a sequence mutually cubic root pairs satisfying $|a_t| \to \infty$ and $b_t \mid a_t^2 + a_t + 1$ for each t (namely, we can apply $\tilde{D}$ to it). If $s := \lim_{t\to\infty} \frac{\log|a_t|}{\log|b_t|} > \frac{3+\sqrt{3}}{2}$ (including ∞), then $(c_t, d_t) = \tilde{T} \circ \tilde{T} \circ \tilde{D}(a_t, b_t)$ is also a

sequence mutually cubic root pairs satisfying $d_t \mid c_t^2 + c_t + 1$ and $|c_t| \to \infty$. Moreover, we have

$$R(s) := \lim_{t\to\infty} \frac{\log|c_t|}{\log|d_t|} = \frac{5s-3}{2s-1}$$

and $\frac{3+\sqrt{3}}{2} < R(s) < s$. □

Proof. (1) Suppose first that (a,b) is a mutually cubic roots pair with $b \mid a^2+a+1$. We clearly have that $(c,d) := (a,(1-a)b)$ satisfy $(1-c) \mid d$ and $c^3 - 1 \equiv_d 0$. Furthermore, we have that $d^3 = (1-a)^3 b^3 \equiv_a b^3 \equiv_a 1$ so that (c,d) is a mutually cubic root pair. On the other hand, if (c,d) is a mutually cubic root pair with $(1-c) \mid d$, then we similarly get that $(c, \frac{d}{1-c})$ is again a mutually cubic root pair satisfying $b \mid a^2+a+1$ and $\tilde{D}(a,b) = (c,d)$.

(2) Assuming that (a_t, b_t) is a mutually cubic root pair satisfying $b_t \mid a_t^2 + a_t + 1$ we get that

$$(c_t,d_t) = \tilde{T}\circ\tilde{T}\circ\tilde{D}(a_t,b_t) = \tilde{T}\circ\tilde{T}(a_t,(1-a_t)b_t) = \tilde{T}\left(\frac{1-a_t^3}{(1-a_t)b_t}, a_t\right)$$
$$= \tilde{T}\left(\frac{a_t^2+a_t+1}{b_t}, a_t\right) = \left(\left(1-\left[\frac{a_t^2+a_t+1}{b_t}\right]^3\right)/a_t, \frac{a_t^2+a_t+1}{b_t}\right)$$

We assumed that $b_t \mid a_t^2+a_t+1$ and $|a_t| \to \infty$ so that in particular $b_t, a_t, 1-a_t \neq 0$ for almost all t and we can divide by them. It follows that (c_t,d_t) are well defined and from part (1) it is also a mutually cubic root pair.

Since $s = \lim_{t\to\infty} \frac{\log|a_t|}{\log|b_t|} > \frac{3+\sqrt{3}}{2} > \frac{1}{2}$ and $|a_t| \to \infty$, we conclude that $\left|\frac{a_t^2+a_t+1}{b_t}\right| \to \infty$ and therefore

$$R(s) := \lim_{t\to\infty} \frac{\log|c_t|}{\log|d_t|} = \frac{5s-3}{2s-1}.$$

It is straight forward to show that if $s > \frac{3+\sqrt{3}}{2}$, then $s > R(s) > \frac{3+\sqrt{3}}{2}$. It follows that since $|d_t| = \left|\frac{a_t^2+a_t+1}{b_t}\right| \to \infty$ we must also have that $|c_t| \to \infty$. Finally, we need to show that $c_t^2 + c_t + 1 \equiv_{b_t} 0$. Indeed, since $\gcd(a_t, d_t) = 1$ we have $c_t \equiv_{d_t} (1-d_t^3)\cdot\frac{1}{a_t} \equiv_{d_t} \frac{1}{a_t}$ so that

$$c_t^2 + c_t + 1 \equiv_{d_t} \frac{1+a_t+a_t^2}{a_t^2} = \frac{1+a_t+a_t^2}{b_t}\frac{b_t}{a_t^2} \equiv_{d_t} 0.$$ ■

Corollary 3.27. There are infinitely many T-orbits in Λ in the open interval $\left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$ and Λ has infinitely many accumulation points. □

Proof. Consider the sequence $(a_t, b_t) = (t, 1)$ of mutually cubic root pairs. This sequence satisfies the conditions from the previous lemma, that is, that $1 = b_t \mid a_t^2 + a_t + 1$ and that $|a_t| = |t| \to \infty$ as $t \to \infty$. Furthermore, we have that $s = \lim \frac{\log|a_t|}{\log|b_t|} = \infty > \frac{3+\sqrt{3}}{2}$ so from the previous lemma, for every n the sequence $[(\tilde{T}^2 \circ \tilde{D})^n(t, 1)]_{t=1}^{\infty}$ is a sequence of mutually cubic root pairs, and it corresponds to the limit $R^n(\infty) \in \Lambda$, where $R(s)$ is defined in the previous lemma. This sequence is decreasing and converges to $\frac{3+\sqrt{3}}{2}$ which is in the segment $\left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$, hence $\frac{3+\sqrt{3}}{2}$ is an accumulation point. Using the fact that Λ is closed under the action of T, we see that there are infinitely many accumulation points in Λ.

Since the only limits of T-orbits are $\frac{3\pm\sqrt{5}}{2}$, it must contain infinitely many orbits in order to have infinitely many accumulation points. ■

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